Today in Physics 218: optical properties of conductors

- ❏ Good and bad conductors
- ❏ Relative phase of *E* and *B* of waves in conductors
- ❏ Reflection from conducting surfaces
- ❏ The characteristic matrix of a conducting layer

False-color infrared image of the Rosebud Nebula, NGC 7129, by Tom Megeath (CfA), Rob Gutermuth and Judy Pipher (UR), using the IRAC instrument on the new Spitzer Space Telescope.

Good and bad conductors

The imaginary part of the wavenumber in a conductor is, as we showed last time,

$$\kappa = \sqrt{\frac{\mu\varepsilon}{2}}\,\frac{\omega}{c}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}-1\right)^{1/2}\;.$$

This is small compared to the real part if the term under the square root is small, which in turn will be true if

$$\sigma \ll \frac{\varepsilon\omega}{4\pi}\qquad\left(\ll \varepsilon\omega = \varepsilon_r\varepsilon_0\omega \text{ in MKS}\right)\;.$$

We would call this the case of a **bad conductor**. Note the frequency dependence: even matter with large conductivity can be a bad conductor for fields that vary rapidly enough in time.

Good and bad conductors (continued)

The opposite, of course, is what we would call a **good conductor**:

$$\sigma \gg \frac{\varepsilon \omega}{4\pi} \quad .$$

Comparing this to the time constant for attenuation of electromagnetic waves as they propagate in conducting media,

$$\tau = \frac{\varepsilon}{2\pi\sigma} \quad ,$$

we see that bad (good) conductors have

$$\tau \gg (\ll) \frac{2}{\omega} = \frac{1}{\pi}\left(\frac{2\pi}{\omega}\right) \quad .$$

Good and bad conductors (continued)

For bad conductors, then,

$$\kappa = \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}-1\right)^{1/2}$$

For $|x| \ll 1$,

$$\sqrt{1+x} \cong 1+x/2:$$

$$\cong \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(1+\frac{1}{2}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2-1\right)^{1/2} = \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\frac{1}{\sqrt{2}}\frac{4\pi\sigma}{\varepsilon\omega}$$

$$= \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}Z \qquad \left(= \frac{\sigma}{2c}\sqrt{\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}} = \frac{\sigma}{2c}Z \text{ in MKS}\right) \quad .$$

Similarly,

$$k \cong \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(1+\frac{1}{2}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2+1\right)^{1/2} \cong \sqrt{\mu\varepsilon}\,\frac{\omega}{c} \gg \kappa \quad .$$

Good and bad conductors (continued)

And for good conductors, since $\left(4\pi\sigma / \varepsilon\omega \right)^2 \gg 1$,

$$\left.\begin{array}{c} k \\ \kappa \end{array}\right\} = \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(\left(1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right)^{1/2}\pm 1\right)^{1/2} \cong \sqrt{\frac{\mu\varepsilon}{2}}\frac{\omega}{c}\left(\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)\pm 1\right)^{1/2}$$

$$= \frac{\sqrt{2\pi\omega\mu\sigma}}{c} \quad \left(= \frac{1}{c}\sqrt{\frac{\omega\mu_r\mu_0\sigma}{2}} \ \text{in MKS}\right) \quad ;$$

$$\tilde{k} = \frac{\sqrt{2\pi\omega\mu\sigma}}{c}\left(1+i\right) \quad .$$

The real and imaginary parts of the complex wavenumber are the same size, in good conductors.

Phases of *E* and *B*

For polarization in the *x* direction, plane waves in conductors look like this:

$$\tilde{\boldsymbol{E}} = \hat{\boldsymbol{x}}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad ,$$

$$\tilde{\boldsymbol{B}} = \hat{\boldsymbol{y}}\tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad .$$

However, it turns out that *E* and *B* are no longer in phase because the wavenumber is complex, as we can show from Faraday's law:

$$\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} = \hat{\boldsymbol{y}}\frac{\partial \tilde{E}_x}{\partial z} = \left(ik - \kappa\right)\hat{\boldsymbol{y}}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$= -\frac{1}{c}\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = \frac{i\omega}{c}\hat{\boldsymbol{y}}\tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \quad ;$$

Phases of E and B (continued)

that is,

$$\tilde{B}_0 = \frac{(k + i\kappa)}{\omega} c\tilde{E}_0 = \frac{c\left|\tilde{k}\right|e^{i\phi}}{\omega}\tilde{E}_0 \qquad \phi = \arctan(\kappa/k)$$

$$\tilde{B} = \hat{y}\,\frac{c\left|\tilde{k}\right|}{\omega}\tilde{E}_0 e^{-\kappa z}e^{i(kz - \omega t + \phi)} \qquad .$$

B is delayed, relative to E. The time-averaged Poynting vector for these out-of-phase waves (Problem 9.20b) is

$$\langle S\rangle = \mathrm{Re}\,\frac{c}{8\pi}\tilde{E}\times\tilde{H}^* = \mathrm{Re}\,\frac{c}{8\pi\mu}\tilde{E}\times\tilde{B}^* = \mathrm{Re}\,\hat{z}\,\frac{c}{8\pi\mu}\frac{c\left|\tilde{k}\right|}{\omega}\left|\tilde{E}_0\right|^2 e^{-2\kappa z}e^{-i\phi}$$

$$= \hat{z}\,\frac{c^2}{8\pi\mu}\frac{\left|\tilde{k}\right|}{\omega}E_0^2 e^{-2\kappa z}\cos\phi = \hat{z}\,\frac{c^2}{8\pi\mu}\frac{k}{\omega}E_0^2 e^{-2\kappa z} \quad . \begin{cases} \hat{z}\,\dfrac{1}{2\mu}\dfrac{k}{\omega}E_0^2 e^{-2\kappa z}\,, \\[2mm] \text{in MKS.} \end{cases}$$

Reflection from conducting surfaces

The boundary conditions for light incident on a conductor aren't different from the ones for nonconductors.

- ❑ If $J = \sigma E,$ then an infinite field would be needed to produce a surface current:

$$J_f = K_f \delta(z) = \sigma E \quad .$$

So we get

$$\varepsilon_1 E_{1\perp} - \varepsilon_2 E_{2\perp} = 4\pi\sigma_f = 0$$

$$B_{1\perp} - B_{2\perp} = 0$$

$$E_{1\parallel} - E_{2\parallel} = 0$$

$$\frac{1}{\mu_1} B_{1\parallel} - \frac{1}{\mu_2} B_{2\parallel} = \frac{4\pi}{c} \left| K_f \times \hat{n} \right| = 0 \quad \text{(still)}.$$

Reflection from conducting surfaces (continued)

Let us therefore consider light incident from a non-conductor onto a conductor, and for simplicity let's restrict ourselves to normal incidence. The fields are

$$\tilde{\boldsymbol{E}}_I = \hat{\boldsymbol{x}}\tilde{E}_{0I}e^{i(k_1 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_I = \hat{\boldsymbol{y}}\sqrt{\mu_1 \varepsilon_1}\,\tilde{E}_{0I}e^{i(k_1 z - \omega t)}$$

$$\tilde{\boldsymbol{E}}_R = \hat{\boldsymbol{x}}\tilde{E}_{0R}e^{i(-k_1 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_R = \hat{\boldsymbol{y}}\sqrt{\mu_1 \varepsilon_1}\,\tilde{E}_{0R}e^{i(-k_1 z - \omega t)}$$

$$\tilde{\boldsymbol{E}}_T = \hat{\boldsymbol{x}}\tilde{E}_{0T}e^{i(\tilde{k}_2 z - \omega t)} \qquad \tilde{\boldsymbol{B}}_T = \hat{\boldsymbol{y}}\frac{c\tilde{k}_2}{\omega}\,\tilde{E}_{0T}e^{i(\tilde{k}_2 z - \omega t)}$$

As usual we put the surface at $z = 0$, and the boundary conditions there give:

Reflection from conducting surfaces (continued)

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\frac{1}{\mu_1}\left(\sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0I} - \sqrt{\mu_1\varepsilon_1}\,\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\frac{c\tilde{k}_2}{\omega}\tilde{E}_{0T} \quad .$$

Note that we've written this in terms of the complex $\tilde{k}_2$; we can still divide through by $\sqrt{\varepsilon_1/\mu_1}$ and rearrange to get

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T} \quad ,$$

where

$$\tilde{\beta} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\frac{c\tilde{k}_2}{\mu_2\omega} \quad .$$

is complex, this time.

Reflection from conducting surfaces (continued)

These equations have the usual solutions,

$$\tilde{E}_{0T} = \frac{2}{1+\tilde{\beta}}\tilde{E}_{0I} \quad , \quad \tilde{E}_{0R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0I} \quad .$$

It is interesting to specialize this for the most common case, that of a good conductor (= an ordinary metal mirror). Here,

$$\tilde{\beta} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\frac{c}{\mu_2\omega}\frac{\sqrt{2\pi\omega\mu\sigma}}{c}(1+i) = \sqrt{\frac{\mu_1\varepsilon_2}{\varepsilon_1\mu_2}}\sqrt{\frac{2\pi\sigma}{\varepsilon_2\omega}}(1+i) \equiv \gamma(1+i)$$

$$= \beta_{\text{non-conductor}}\sqrt{\frac{2\pi\sigma}{\varepsilon_2\omega}}(1+i) \quad , \text{ where}$$

$$\gamma = \sqrt{\frac{\mu_1}{\varepsilon_1\mu_2}}\sqrt{\frac{2\pi\sigma}{\omega}} \quad \left(= \sqrt{\frac{\mu_1}{\varepsilon_1\mu_2}}\sqrt{\frac{\sigma}{2\omega}} \text{ in MKS}\right).$$

Reflection from conducting surfaces (continued)

❑ If σ were infinite, $\tilde{\beta}$ would be too, so

$$\tilde{E}_{0T} = 0 \quad , \quad \tilde{E}_{0R} = -\tilde{E}_{0I} \quad .$$

No transmission, and reflected light 180° out of phase (upside-down) with respect to the incident light.

❑ If σ were merely large (as in Problem 9.21),

$$\frac{I_R}{I_I} = \frac{\left|\tilde{E}_{0R}\right|^2}{\left|\tilde{E}_{0I}\right|^2} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*} = \frac{1-\left(\tilde{\beta}+\tilde{\beta}^*\right)+\tilde{\beta}\tilde{\beta}^*}{1+\left(\tilde{\beta}+\tilde{\beta}^*\right)+\tilde{\beta}\tilde{\beta}^*}$$

$$= \frac{1-(2\gamma)+2\gamma^2}{1+(2\gamma)+2\gamma^2} = 0.923 \quad \text{for aluminum at } \lambda=0.55 \ \mu\text{m} \quad .$$

Reflection and transmission by conducting layers

Consider a plane-parallel metal layer, with conductivity σ and thickness d, bounded on either side by nonconducting media (see next slide). What are the transmittance and reflectance of this structure?

❑ For simplicity, let's stick to **normal incidence**. The boundary conditions at the first surface are the ones we already used for a single surface, plus an extra reflected wave:

$$\tilde{E}_{\|,1} = \tilde{E}_{0I} + \tilde{E}_{0R1} = \tilde{E}_{0T1} + \tilde{E}_{0R2} \quad ,$$

$$\tilde{H}_{\|,1} = \frac{1}{\mu_0}\sqrt{\mu_0 \varepsilon_0}\left(\tilde{E}_{0I} - \tilde{E}_{0R1}\right) = \frac{1}{\mu_1}\frac{c\tilde{k}_1}{\omega}\left(\tilde{E}_{0T1} - \tilde{E}_{0R2}\right) \quad .$$

Reflection and transmission by conducting layers (continued)
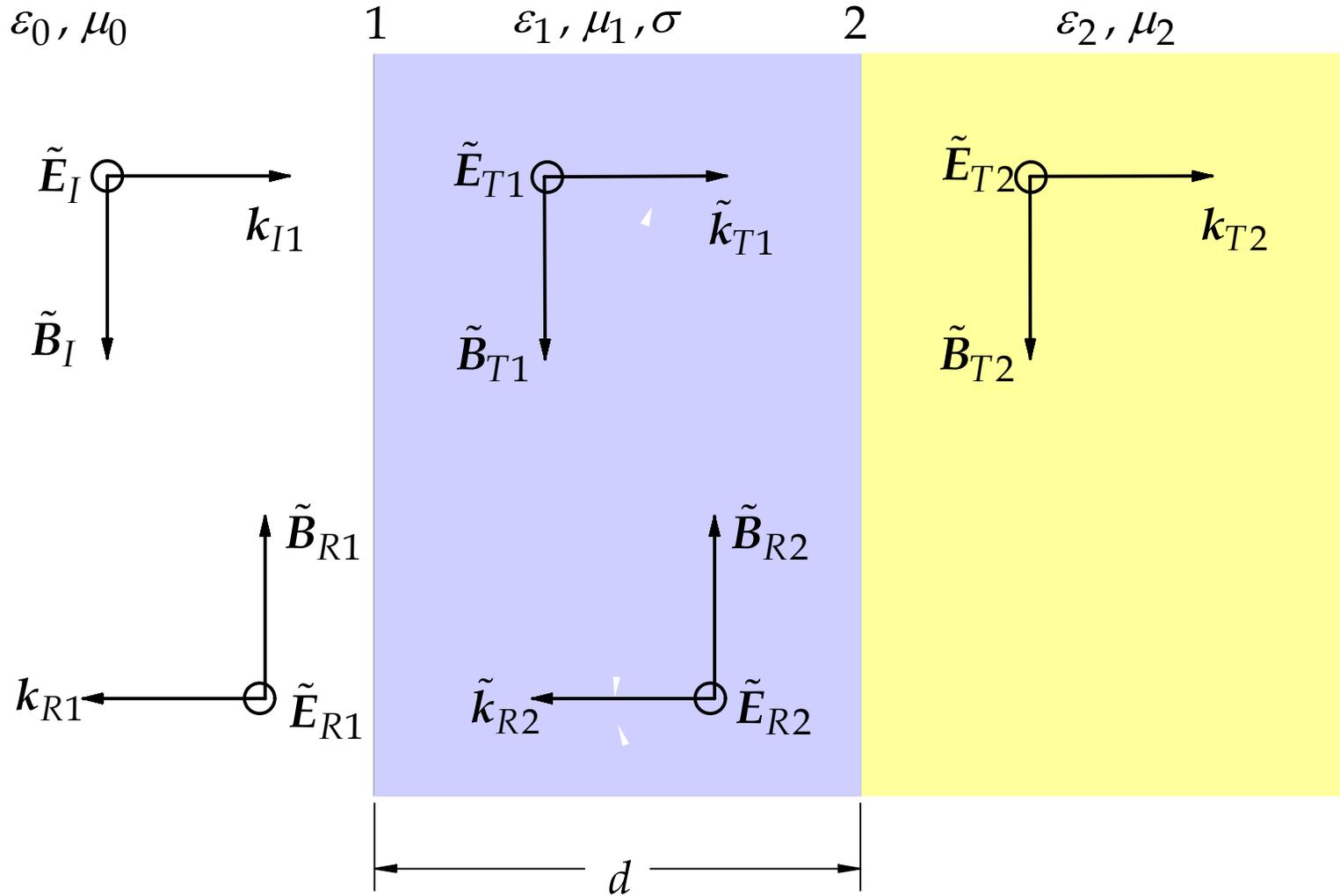
Reflection and transmission by conducting layers (continued)

❑ As a wave crosses the layer it travels a distance d. Compared to the undisplaced wave that would have resulted if the slab were not there, it undergoes a phase change of

$$\delta_1 = \tilde{k}_1 d \quad .$$

❑ Thus the $E_{\parallel}$ and $H_{\parallel}$ boundary conditions at surface 2 are

$$\tilde{E}_{\parallel,2} = \tilde{E}_{0T1} e^{i\delta_1} + \tilde{E}_{0R2} e^{-i\delta_1} = \tilde{E}_{0T2} e^{i\delta_1} \quad ,$$

$$\tilde{H}_{\parallel,2} = \frac{1}{\mu_1} \frac{c\tilde{k}_1}{\omega} \left(\tilde{E}_{0T1} e^{i\delta_1} - \tilde{E}_{0R2} e^{-i\delta_1} \right) = \sqrt{\frac{\varepsilon_2}{\mu_2}} \tilde{E}_{0T2} e^{i\delta_1}$$

Reflection and transmission by conducting layers (continued)

❑ By now, this probably looks familiar: the boundary conditions, and the expressions for $E_\parallel$ and $H_\parallel$ at the surfaces, are just the same as in the case of nonconductors, with

$$Y_1 = \sqrt{\frac{\varepsilon_1}{\mu_1}} \quad \rightarrow \quad \frac{1}{\mu_1}\frac{c\tilde{k}_1}{\omega} \quad ,$$

$$\delta_1 = \frac{2\pi n_1 d}{\lambda} \quad \rightarrow \quad \tilde{k}_1 d \quad .$$

Thus the conducting layer has a characteristic matrix that looks very similar to that of a nonconducting layer. The major differences are that the admittance Y and the phase shift δ are complex, not real.

Characteristic matrix of a conducting layer at normal incidence

With the definitions

$$Y_1 = \frac{1}{\mu_1} \frac{c\tilde{k}_1}{\omega} \quad \text{and} \quad \delta_1 = \tilde{k}_1 d \quad,$$

where

$$\tilde{k}_1 = \begin{cases} \dfrac{\sqrt{2\pi\omega\mu_1\sigma_1}}{c}(1+i) & \text{good conductor} \\[4mm] \sqrt{\mu_1\varepsilon_1}\,\dfrac{\omega}{c} + i\dfrac{2\pi\sigma_1}{c}\sqrt{\dfrac{\mu_1}{\varepsilon_1}} & \text{bad conductor} \end{cases} \quad,$$

the characteristic matrix is as usual

$$M_1 = \begin{bmatrix} \cos\delta_1 & -i\sin\delta_1 / Y_1 \\ -iY_1\sin\delta_1 & \cos\delta_1 \end{bmatrix} \quad.$$

Characteristic matrix of a conducting layer at normal incidence (continued)

This can be used in transmittance and reflectance calculations just as before, in combination with other layers, conducting or non conducting. In particular, we still can use

$$t = \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0 Y_{p+1} + m_{21} + m_{22}Y_{p+1}} \quad ,$$

$$r = \frac{m_{11}Y_0 + m_{12}Y_0 Y_{p+1} - m_{21} - m_{22}Y_{p+1}}{m_{11}Y_0 + m_{12}Y_0 Y_{p+1} + m_{21} + m_{22}Y_{p+1}} \quad ,$$

$$\rho = \frac{\langle S_{R1,\perp} \rangle}{\langle S_{I,\perp} \rangle} = |r|^2 \quad \text{and} \quad \tau = \frac{\langle S_{T,p+1,\perp} \rangle}{\langle S_{I,\perp} \rangle} = \frac{Y_{p+1}}{Y_0}|t|^2 \quad .$$

Note, however, that $\rho + \tau$ no longer makes 1; there is **absorption** in the conducting layer, due to its resistance.