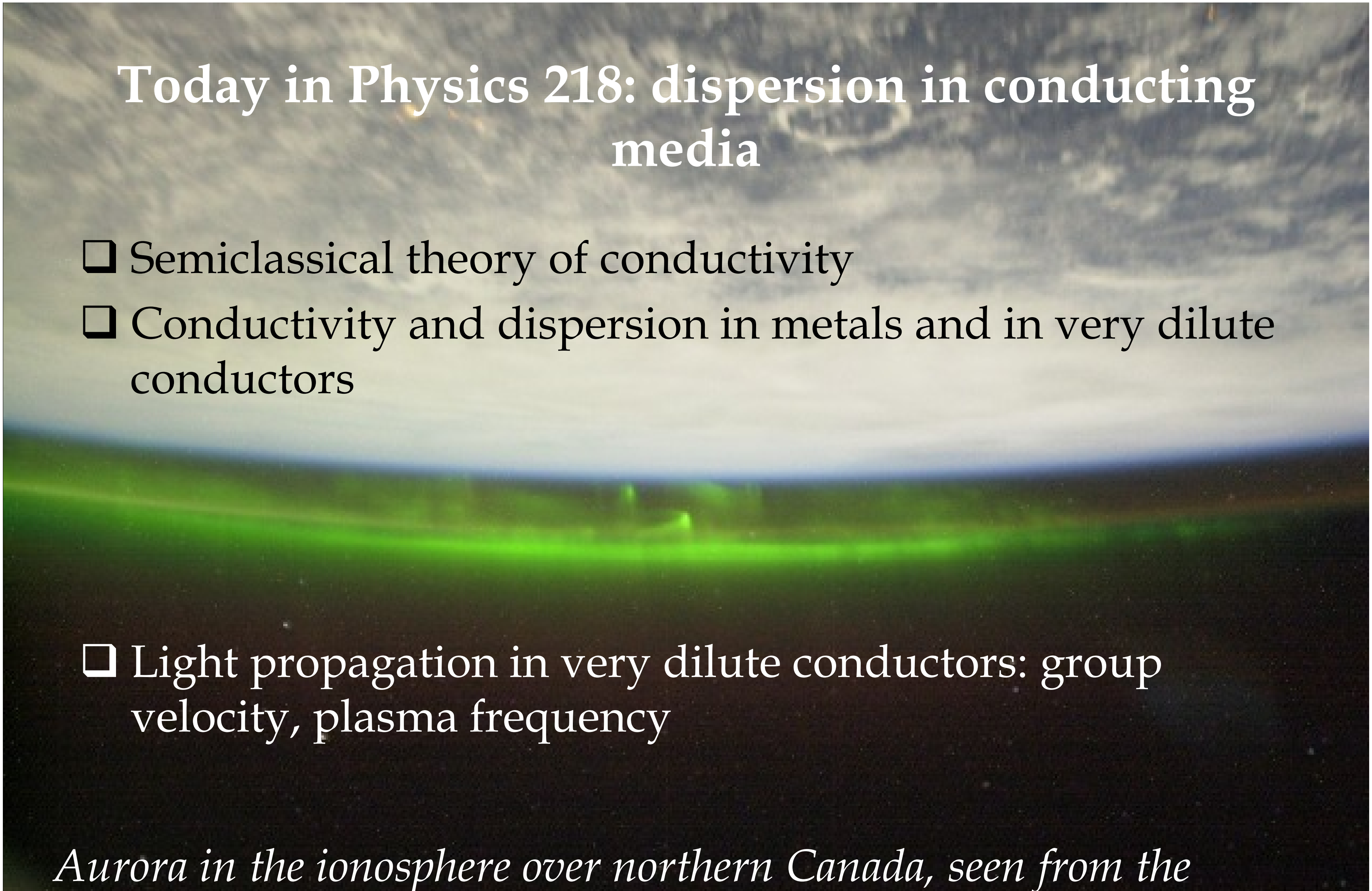
Today in Physics 218: dispersion in conducting media

- Semiclassical theory of conductivity
- Conductivity and dispersion in metals and in very dilute conductors
- Light propagation in very dilute conductors: group velocity, plasma frequency



Aurora in the ionosphere over northern Canada, seen from the International Space Station. Photo by Don Pettit.

Semiclassical theory of conductivity

On microscopic scales, conductors differ from dielectrics mostly in that many of the electrons are free: there are no "springs and shock absorbers" binding them to their host atoms or molecules.

- Thus the net force on an electron at $x = 0$, interacting with a wave of light and its electric field $\boldsymbol{E} = \boldsymbol{E}_0 e^{-i\omega t}$ is

$$F = qE - m_e \gamma_0 \frac{dx}{dt} = m_e a = m_e \frac{d^2 x}{dt^2} \quad ,$$

  and the equation of motion is

$$\frac{d^2 x}{dt^2} + \gamma_0 \frac{dx}{dt} = \frac{q}{m_e} E_0 e^{-i\omega t} \quad ,$$

Semiclassical theory of conductivity (continued)

for which the solution is obtained easily, in the manner employed last lecture:

$$x = x_0 e^{-i\omega t}, \quad x_0 = -\frac{\dfrac{q}{m_e} E_0}{\omega^2 + i\omega\gamma_0} \quad .$$

- The damping is provided by our old friend, Ohmic resistance, and if the conducting medium is isotropic we will need to refer to only one damping constant γ_0 rather than a whole host of γ_j.
- In turn, γ_0 will be larger or smaller if the rate at which electrons suffer collisions is larger or smaller , as we'll see in a moment.

Semiclassical theory of conductivity (continued)

- Now, under the influence of the passing wave, the electron moves with speed $v = dx/dt$.
- Suppose that the medium around the electron consists of N atoms per unit volume, each of which contributes f_0 free electrons. Then the mobile charge density is

$$\rho_{\text{mobile}} = Nf_0 q \quad ,$$

whence the current density is

$$J = \rho_{\text{mobile}} v = Nf_0 q \frac{dx}{dt} = -Nf_0 q(-i\omega)\frac{\frac{q}{m_e}E_0 e^{-i\omega t}}{\omega^2 + i\gamma_0\omega}$$

$$= \frac{Nf_0 q^2}{m_e}\frac{1}{\gamma_0 - i\omega}E \quad .$$

Semiclassical theory of conductivity (continued)

- But $\boldsymbol{J} = \sigma \boldsymbol{E}$, so we have obtained here another expression for the conductivity:

$$\boxed{\sigma = \frac{N f_0 q^2}{m_e} \frac{1}{\gamma_0 - i\omega} \quad .}$$

- There are two useful limiting cases to distinguish:
  - **metals** (liquid or solid), for which N is large and the rate of collisions of conduction electrons and ions is also large, so that γ_0 is large.
  - **conducting gases**, for which N and γ_0 are much smaller.

Conductivity in metals

- In metals, $\gamma_0 \gg \omega$, and the real part of the conductivity is much larger than the imaginary part:

$$\sigma = \frac{Nf_0 q^2}{m_e} \frac{1}{\sqrt{\gamma_0^2 + \omega^2}} e^{i \arctan \omega/\gamma_0} \cong \frac{Nf_0 q^2}{m_e \gamma_0} \quad .$$

- Compare this with the expression we derived in PHY 217:

$$\sigma = \frac{nq^2}{m_e} \bar{t} = \frac{Nf_0 q^2}{m_e} \bar{t} \quad ,$$

where n was the number of electrons per unit volume and $\bar{t}$ was the average time between collisions. Evidently, γ_0 is the average *rate* of electron collisions:

$$\gamma_0 = \frac{1}{\bar{t}} \quad .$$

Flashback: collisions, drift velocity and conductivity (PHY 217, 18 November 2002)

Consider a chunk of metal with N mobile charges in it and an electric field $\boldsymbol{E}$ present. If it has been a time t_i since particle i last suffered a collision, and if it left that collision at speed v_i, then the momentum of this particle is

$$\boldsymbol{p}_i = m\boldsymbol{v}_i + q\boldsymbol{E}t_i \quad .$$

So a snapshot of the metal would reveal an average value of carrier momentum given by

$$\overline{\boldsymbol{p}} = m\overline{\boldsymbol{v}} = \frac{1}{N}\sum_{i=i}^{N}\boldsymbol{p}_i = \frac{1}{N}\sum_{i=1}^{N}\left(m\boldsymbol{v}_i + q\boldsymbol{E}t_i\right) \quad .$$

The “starting” speeds v_i are endowed by collisions with the fixed ions; their energy, in turn, comes from the *thermal* energy (heat) of the medium.

Flashback: collisions, drift velocity and conductivity (continued)

Thermal motions in a solid or liquid are (essentially) random in magnitude and direction, so if N is large,

$$\sum_{i=1}^{N} m\boldsymbol{v}_i = 0 \quad ; \text{ thus,}$$

$$\overline{\boldsymbol{v}} = \frac{q\boldsymbol{E}}{mN}\sum_{i=1}^{N} t_i = \frac{q\boldsymbol{E}}{m}\overline{t} \quad ,$$

Drift velocity

$$\text{and} \quad \boldsymbol{J} = \rho\overline{\boldsymbol{v}} = nq\overline{\boldsymbol{v}} = \boxed{\frac{nq^2\overline{t}}{m}\boldsymbol{E} \equiv \sigma\boldsymbol{E}} \quad ,$$

where n is the *number* density of carriers. Clearly $\boldsymbol{J}$ should be linear in $\boldsymbol{E}$ if the carrier velocities are mostly thermal, and if collisions take place.

Conductivity in very dilute conductors

- In the limit of small damping, $\gamma_0 \ll \omega$, the conductivity is purely imaginary:

$$\sigma \cong i \frac{N f_0 q^2}{m_e \omega} \quad .$$

This is what you get in gases, under two conditions that are worth distinguishing:

- **ionized gases**, in which collisions (close interactions of pairs of charges) still provide most of the resistance.
- **plasmas**, ionized gases in which collisions are so infrequent that the motions of electrons are *collective*: the fields of all the charges in an electron's neighborhood are important in determining its motion.

Light propagation in very dilute conductors

- Let's start with the expression for the complex wavenumber that we got directly from the wave equation for the fields in conducting media (lecture, 13 February):

$$\tilde{k}^2 = \mu\varepsilon\frac{\omega^2}{c^2} + i\frac{4\pi\sigma\omega\mu}{c^2} \quad .$$

- Assume that the gas is sufficiently rarefied to take $\mu = \varepsilon = 1$; then,

$$\tilde{k}^2 = \frac{\omega^2}{c^2} + i\frac{4\pi\omega}{c^2}\left(i\frac{Nf_0q^2}{m_e\omega}\right)$$

$$= \frac{\omega^2}{c^2}\left(1 - \frac{4\pi Nf_0q^2}{m\omega^2}\right) = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = k^2 \quad ,$$

Light propagation in very dilute conductors (continued)

- where we have defined the (angular) **plasma frequency**:

$$\omega_p \equiv \sqrt{\frac{4\pi N f_0 q^2}{m}} \quad .$$

Dilute conductors provide us with the simplest example of dispersion (which, you'll remember, is a variation of refractive index with frequency). We shall now consider two new features of dispersion, using this example.

- **Wave speed.** With $k = n\omega/c$, the phase velocity of light in a dilute conductor is

$$v = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{1-\left(\omega_p/\omega\right)^2}} > c \quad (!!!).$$

Light propagation in very dilute conductors (continued)

- Now, we experts in the special theory of relativity all know that cause-and-effect relationships can change no faster than by propagation at the speed of light in vacuum, *c*. But
  - any *change* of this wave, such as would constitute a signal, would have to be due to a change in the wave's amplitude or energy, and
  - such changes turn out *not* to propagate at the phase velocity.
- To see this, consider broadcasting a plane wave for which we have somehow *modulated* the amplitude sinusoidally at some angular frequency $\omega_{\mathrm{mod}} \ll \omega$:

Light propagation in very dilute conductors (continued)

$$\tilde{E}(z,t) = \tilde{E}_0(z,t)\, e^{i(kz-\omega t)} = E_{0,\mathrm{mod}}\, e^{i(k_{\mathrm{mod}} z - \omega_{\mathrm{mod}} t)} e^{i(kz-\omega t)} \quad .$$

- The peaks and troughs of the wave itself move of course at $v = \omega/k$, but the peaks and troughs of the modulation move so as to keep

$$k_{\mathrm{mod}} z - \omega_{\mathrm{mod}} t = \text{ constant} \quad ;$$

That is,

$$k_{\mathrm{mod}} dz - \omega_{\mathrm{mod}} dt = 0 \quad .$$

$$v_g \equiv \frac{dz}{dt} = \frac{\omega_{\mathrm{mod}}}{k_{\mathrm{mod}}} \quad .$$

- We don't know *a priori* what k_{mod} is. But we do know ω as a function of *k*, and can write

$$\omega_{\mathrm{mod}} = \omega' - \omega = \omega(k') - \omega(k) \quad , \quad k_{\mathrm{mod}} = k' - k \quad .$$

Light propagation in very dilute conductors (continued)

- And since $\omega_{\text{mod}} \ll \omega$, we can express ω' as a Taylor series about ω, and neglect all but the first two terms:

$$v_g = \frac{\omega' - \omega}{k' - k} = \frac{1}{k' - k}\left[\left(\omega + \frac{d\omega}{dk}(k' - k) + \ldots\right) - \omega\right]$$

$$\cong \frac{d\omega}{dk} \quad . \qquad \text{Group velocity}$$

- For our dilute conductor,

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left(\frac{d}{d\omega}\frac{\omega}{c}\sqrt{1 - \omega_p^2/\omega^2}\right)^{-1}$$

$$= c\left(\frac{d}{d\omega}\left(\omega^2 - \omega_p^2\right)^{1/2}\right)^{-1} = c\left(\frac{1}{2}\left(\omega^2 - \omega_p^2\right)^{-1/2} 2\omega\right)^{-1}$$

Light propagation in very dilute conductors (continued)

$$v_g = \frac{c}{\omega}\left(\omega^2 - \omega_p^2\right)^{1/2} = c\sqrt{1-\omega_p^2 \big/ \omega^2} < c \quad (!!!!).$$

- Thus intensity changes, and other signals, travel at a speed smaller than *c*, even though the waves that make up the signal travel faster than *c*. Relativity still works.
- Note that for nondispersive media,

$$\frac{\omega}{k} = \frac{d\omega}{dk} = \frac{c}{n} < c \quad \text{, always.}$$

Light propagation in very dilute conductors (continued)

- **Opacity for $\omega < \omega_p$.** For angular frequencies less than the plasma frequency,

$$1 - \omega_p^2 / \omega^2 < 0 \quad ,$$

so the square root of this quantity is purely imaginary:

$$k = \frac{\omega}{c}\sqrt{1 - \omega_p^2 / \omega^2} = \frac{i}{c}\sqrt{\omega_p^2 - \omega^2} \quad ,$$

and waves are strongly attenuated in this medium:

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(kz - \omega t)} = \tilde{\boldsymbol{E}}_0 e^{-\frac{z}{c}\sqrt{\omega_p^2 - \omega^2}} e^{-i\omega t} \quad ;$$

the medium is **opaque.**

Light propagation in very dilute conductors (continued)

- This strong attenuation is a manifestation of a very large resistive part to the medium's impedance. Since vacuum has only a non-resistive impedance, the mismatch between vacuum and plasma would result in strong *reflection* by the plasma of light at this frequency ($\omega < \omega_p$).
- A good example of a plasma is the **ionosphere**: the partially-ionized upper reaches (50-1000 km) of Earth's atmosphere. Here the free electron density is of order $Nf_0 = 10^5\ \text{cm}^{-3}$, for a plasma frequency of

$$\nu_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{4\pi N f_0 q^2}{m_e}} = 3\times 10^6\ \text{Hz} = 3\ \text{MHz}.$$

Light propagation in very dilute conductors (continued)

- Thus the ionosphere is a good reflector for AM radio frequencies (500 kHz – 1.6 MHz), but a poor reflector for FM (87 MHz – 110 MHz). This explains several phenomena familiar to those who still listen to the radio:
  - **AM radio stations are long-ranged, and FM stations aren't.** That's because an AM signal from over the horizon can reflect off the ionosphere and get to your radio; the FM signal just keeps going off into space.
  - **Intense solar activity ruins AM radio reception.** Solar flares bombard Earth with lots of ionized gas, severly disrupting the ionosphere and changing the plasma frequency all over the place. Thus the reflection, and AM reception, are inconsistent.

Light propagation in very dilute conductors (continued)

- **You can receive very distant AM stations at night that you can't get during the day**. The ionization is produced by sunlight. At night the electrons and ions recombine. This recombination takes place much faster the denser the gas is. The gas is denser lower in the atmosphere, so recombination proceeds from the bottom up – the reflecting surface retreats to higher altitude at night. Thus a single reflection can reach you from farther away. When the sunlight comes back, the edge of the ionosphere returns to the lower altitude.

Too bad nobody listens to the radio any more…