Today in Physics 218: guided waves

- Metallic waveguides
- Light propagation in hollow conductive waveguides
- The TE modes of rectangular metal waveguides

X-band (ν = 7-12 GHz) horn, reduced-height waveguide and magnetron (Malcom Strandberg, MIT; Fritz Goro, Life Magazine *photo.)*

Metallic waveguides

Soon after powerful sources of coherent high-frequency radio and microwave radiation were invented, in the decade before World War II, it became common for experimenters and engineers to use **waveguides** to "pipe" the radiation around, instead of using free-space propagation, lenses and mirrors.

- It might seem obvious that this could work well: the light is confined by the metal pipe, and propagates at high efficiency *via* high-incidence reflections.
- Convenient, too: standard components were manufactured, that fit together like plumbing fixtures.

Metallic waveguides (continued)

Malcom Strandberg (MIT) demonstrates collection and transmission through serpentine waveguide of microwave power large enough to light up the neon bulb he's holding. (Fritz Goro, *Life Magazine* photo, 1945.)

Metallic waveguides (continued)

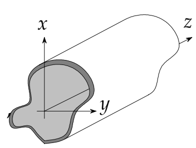
- That's not all there is to it, though; the waveguides are comparable to or smaller than the wavelength of the light propagating through them, and restrict the light's properties in ways different from free space.
- The War brought an explosion in the development of microwave engineering for radar applications. virtually all the top American physicists not already involved in the Manhattan Project – notably at the MIT Radiation Laboratory – worked feverishly to invent a wide variety of waveguide-based devices, generating improved radars, postwar industrial applications, and many frightfully elegant E&M problems. (See Jackson, chapter 8.)

Light propagation in hollow conductive waveguides

We want first to deal with the question, how do monochromatic plane waves propagate in metal pipes?

Let us suppose that waves of the form

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)} \quad , \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(kz-\omega t)} \quad ,$$



will propagate, where in general

$$\tilde{\boldsymbol{E}}_0 = \tilde{E}_x(x,y)\hat{\boldsymbol{x}} + \tilde{E}_y(x,y)\hat{\boldsymbol{y}} + \tilde{E}_z(x,y)\hat{\boldsymbol{z}}$$

$$\tilde{\boldsymbol{B}}_0 = \tilde{B}_x(x,y)\hat{\boldsymbol{x}} + \tilde{B}_y(x,y)\hat{\boldsymbol{y}} + \tilde{B}_z(x,y)\hat{\boldsymbol{z}} \quad .$$

In choosing this form we anticipate that the confinement may prevent constant-amplitude wavefronts, and may not involve transverse waves. Let us now apply Maxwell's equations, for the vacuum within the pipe, to the fields (Problem !9.26a):

Light propagation in hollow conductive waveguides (continued)

Faraday's law gives: $\nabla \times \tilde{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial \tilde{\boldsymbol{B}}}{\partial t}$,

$$\left(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\right)\hat{\boldsymbol{z}} = \frac{i\omega}{c}\tilde{\boldsymbol{B}} \quad ,$$

$$\left(\frac{\partial \tilde{E}_{0z}}{\partial y} - ik\tilde{E}_{0y}\right)\hat{\boldsymbol{x}} + \left(ik\tilde{E}_{0x} - \frac{\partial \tilde{E}_{0z}}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial \tilde{E}_{0y}}{\partial x} - \frac{\partial \tilde{E}_{0x}}{\partial y}\right)\hat{\boldsymbol{z}} = \frac{i\omega}{c}\tilde{\boldsymbol{B}}_0 \quad ,$$

and similarly Ampère's law gives $\nabla \times \tilde{\boldsymbol{B}} = \frac{1}{c}\frac{\partial \tilde{\boldsymbol{E}}}{\partial t}$,

$$\left(\frac{\partial \tilde{B}_{0z}}{\partial y} - ik\tilde{B}_{0y}\right)\hat{\boldsymbol{x}} + \left(ik\tilde{B}_{0x} - \frac{\partial \tilde{B}_{0z}}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial \tilde{B}_{0y}}{\partial x} - \frac{\partial \tilde{B}_{0x}}{\partial y}\right)\hat{\boldsymbol{z}} = -\frac{i\omega}{c}\tilde{\boldsymbol{E}}_0 .$$

Light propagation in hollow conductive waveguides (continued)

Now, it turns out to be possible to solve this system of (six!) equations for the x and y components of $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of their z components. For instance, consider the y component of Faraday's law and the x component of Ampère's law:

$$ik\tilde{E}_{0x} - \frac{\partial \tilde{E}_{0z}}{\partial x} = \frac{i\omega}{c}\tilde{B}_{0y} \quad , \quad \frac{\partial \tilde{B}_{0z}}{\partial y} - ik\tilde{B}_{0y} = -\frac{i\omega}{c}\tilde{E}_{0x} \quad .$$

Multiply the first one by k, the second by ω/c, and subtract:

$$ik^2\tilde{E}_{0x} - k\frac{\partial \tilde{E}_{0z}}{\partial x} - \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y} + \frac{i\omega k}{c}\tilde{B}_{0y} = \frac{i\omega k}{c}\tilde{B}_{0y} + \frac{i\omega^2}{c^2}\tilde{E}_{0x} \quad ,$$

or $$ik^2\tilde{E}_{0x} - \frac{i\omega^2}{c^2}\tilde{E}_{0x} = k\frac{\partial \tilde{E}_{0z}}{\partial x} + \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y} \quad ,$$

Light propagation in hollow conductive waveguides (continued)

or even $$\tilde{E}_{0x} = \frac{i}{\dfrac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{E}_{0z}}{\partial x} + \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y}\right) \quad .$$

I'll spare you the tedium of the other three steps and cut to the answers:

$$\tilde{E}_{0y} = \frac{i}{\dfrac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{E}_{0z}}{\partial y} - \frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial x}\right) \quad ,$$

$$\tilde{B}_{0x} = \frac{i}{\dfrac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{B}_{0z}}{\partial x} - \frac{\omega}{c}\frac{\partial \tilde{E}_{0z}}{\partial y}\right) \quad ,$$

Light propagation in hollow conductive waveguides (continued)

and $$\tilde{B}_{0y} = \frac{i}{\dfrac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{B}_{0z}}{\partial y} + \frac{\omega}{c}\frac{\partial \tilde{E}_{0z}}{\partial x}\right) \quad .$$

- Note that if the waves are transverse $\left(\tilde{B}_{0z} = \tilde{E}_{0z} = 0\right)$, as they are in free space, then they are particularly boring: all of the x and y components of the field amplitudes are zero too, and there is no wave.

It also turns out that if these four results of the Faraday and Ampère laws are inserted into the other two Maxwell equations, the result will be uncoupled equations for E_z and B_z. To wit (problem !9.26b):

Light propagation in hollow conductive waveguides (continued)

$$\nabla \cdot \tilde{E} = 0$$

$$\frac{\partial \tilde{E}_{0x}}{\partial x} e^{i(kz-\omega t)} + \frac{\partial \tilde{E}_{0y}}{\partial y} e^{i(kz-\omega t)} + ik\tilde{E}_{0z} e^{i(kz-\omega t)} = 0$$

$$\frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial^2 \tilde{E}_{0z}}{\partial x^2} + \cancel{\frac{\omega}{c}\frac{\partial^2 \tilde{B}_{0z}}{\partial x \partial y}} + k\frac{\partial^2 \tilde{E}_{0z}}{\partial y^2} - \cancel{\frac{\omega}{c}\frac{\partial^2 \tilde{B}_{0z}}{\partial y \partial x}} \right) + ik\tilde{E}_{0z} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2 \right) \tilde{E}_{0z} = 0 \quad .$$

Similarly, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2 \right) \tilde{B}_{0z} = 0 \quad .$$

Light propagation in hollow conductive waveguides (continued)

Of course, there are still boundary conditions to apply. In order to solve these second-order differential equations, we need two boundary conditions, chosen from the usual suspects:

$$\varepsilon_1 E_{\perp,1} - \varepsilon_2 E_{\perp,2} = 4\pi\sigma_f \qquad B_{\perp,1} - B_{\perp,2} = 0$$

$$\mathbf{E}_{\parallel,1} - \mathbf{E}_{\parallel,2} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_{\parallel,1} - \frac{1}{\mu_2}\mathbf{B}_{\parallel,2} = \frac{4\pi}{c}\mathbf{K}_f \times \hat{\mathbf{n}} \quad ,$$

which in the case of *perfectly* conducting walls (for which **E** = 0 and **B** = 0 inside the conductor) and vacuum inside become

$$E_{\perp,1} = 4\pi\sigma_f \qquad B_{\perp,1} = 0$$

$$\mathbf{E}_{\parallel,1} = 0 \qquad \mathbf{B}_{\parallel,1} = \frac{4\pi}{c}\mathbf{K}_f \times \hat{\mathbf{n}} \quad .$$

Light propagation in hollow conductive waveguides (continued)

We can use *any* two. Clearly the two that will be the simplest to use are the ones without source terms:

$$\mathbf{E}_{\parallel,1} = 0 \quad , \quad B_{\perp,1} = 0 \quad .$$

TE modes of rectangular metal waveguides

Thus there are two special cases to consider:

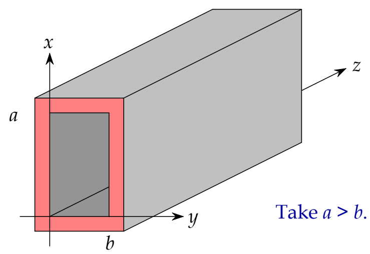
$\tilde{E}_{0z} = 0$, TE waves

$\tilde{B}_{0z} = 0$. TM waves

As we've seen, there are no TEM waves in the hollow conductive waveguide. The wave solutions to the Maxwell equations in hollow waveguides have nonzero longitudinal components to ***E*** or ***B*** or both.

- Let's pursue one of these, and derive the wave solution in a concrete example: TE waves in a rectangular waveguide, with dimensions *a* and *b*.

TE modes of rectangular metal waveguides (continued)



- Considering our previous work, all we have to do is solve for $\tilde{B}_{0z}$. Let's try it with separation of variables.

TE modes of rectangular metal waveguides (continued)

- To solve:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right)\tilde{B}_{0z} = 0 \quad .$$

Set $\tilde{B}_{0z}(x,y) = X(x)Y(y)$, and divide through by XY:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{\omega^2}{c^2} - k^2 = 0 \quad , \text{ or}$$

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0 \quad , \text{ which separates into}$$

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + k_y^2 Y = 0 \quad .$$

TE modes of rectangular metal waveguides (continued)

- The general solution for the *x* part is, of course,

$$X(x) = A\sin k_x x + C\cos k_x x \quad .$$

The boundary conditions requires that $\tilde{B}_{0x}$ vanish at $x = 0$ and *a*. Now, our function *X* is part of $\tilde{B}_{0z}$, not $\tilde{B}_{0x}$. But a few minutes ago we proved that

$$\tilde{B}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k\frac{\partial \tilde{B}_{0z}}{\partial x} - \frac{\omega}{c}\frac{\partial \tilde{E}_{0z}}{\partial y}\right) = \frac{ikY}{\frac{\omega^2}{c^2} - k^2}\frac{dX}{dx} \quad ,$$

so we apply

$$\frac{dX}{dx}(0) = \frac{dX}{dx}(a) = 0 \quad .$$

TE modes of rectangular metal waveguides (continued)

- Thus,

$$\frac{dX}{dx}(x) = k_x A\cos k_x x - k_x C\sin k_x x \quad ,$$

$$\frac{dX}{dx}(0) = k_x A = 0 \quad \Rightarrow \quad A = 0 \quad ,$$

$$\frac{dX}{dx}(a) = -k_x C\sin k_x a = 0 \quad \Rightarrow \quad k_x a = m\pi \quad , \quad m = 0,1,2,\ldots$$

- Similarly, for the *y* part of $\tilde{B}_{0z}$, we get

$$Y(y) = C\cos\frac{n\pi y}{b} \quad , \quad n = 0,1,2,\ldots$$

TE modes of rectangular metal waveguides (continued)

- So the solution is

$$\tilde{B}_{0z} = B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad ,$$

$$m,n = 0,1,2,\ldots$$

Longitudinal ***B*** for TE modes (TE_{mn})

and all the rest of the components of ***E*** and ***B*** can be worked out from the expressions derived earlier (pages 7-9),

$$\tilde{E}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y} \quad , \quad \tilde{E}_{0y} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial x} \quad ,$$

TE modes of rectangular metal waveguides (continued)

and

$$\tilde{B}_{0x} = \frac{ik}{\frac{\omega^2}{c^2} - k^2} \frac{\partial \tilde{B}_{0z}}{\partial x} \quad , \quad \tilde{B}_{0y} = \frac{ik}{\frac{\omega^2}{c^2} - k^2} \frac{\partial \tilde{B}_{0z}}{\partial y} \quad .$$

- One more nuance of the solution needs to be mentioned: it turns out that the m,n = 0,0 mode cannot occur (Problem 9.27). This mode would actually be a TEM mode (that is, it would have $\tilde{B}_{0z} = 0$ as well as $\tilde{E}_{0z} = 0$), which as we have already noted cannot propagate in hollow conducting waveguides.