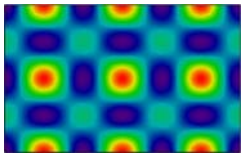
Today in Physics 218: more on waveguides and other transmission lines

- Waveguide modes
- Dispersion and cut-off in waveguides
- Massive photons?
- The real reason there are no TEM modes in hollow conducting waveguides
- TEM modes in coaxial waveguides



Intensity of the TE_{32} mode in rectangular waveguides. Red is the highest intensity, darker blues approach zero, and the other rainbow colors represent intermediate intensity values.

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Waveguide modes: intensity in rectangular waveguides

Let's calculate the intensity within the waveguide for the TE mode we worked out last time.

The time-averaged Poynting vector, $\langle \boldsymbol{S} \rangle = \frac{c}{8\pi} \boldsymbol{E} \times \boldsymbol{B}^*$, is

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\left[\left(E_y B_z^* - \cancel{E_z B_y^*}\right)\hat{\boldsymbol{x}} + \left(\cancel{E_z B_x^*} - E_x B_z^*\right)\hat{\boldsymbol{y}} + \left(E_x B_y^* - E_y B_x^*\right)\hat{\boldsymbol{z}}\right] .$$

Last time we showed that

$$\tilde{E}_{0x} = \frac{i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial y} \quad , \quad \tilde{E}_{0y} = \frac{-i}{\frac{\omega^2}{c^2} - k^2}\frac{\omega}{c}\frac{\partial \tilde{B}_{0z}}{\partial x} \quad ,$$

$$\tilde{B}_{0x} = \frac{-ik}{\frac{\omega^2}{c^2} - k^2}\frac{\partial \tilde{B}_{0z}}{\partial x} \quad \text{, and} \quad \tilde{B}_{0y} = \frac{ik}{\frac{\omega^2}{c^2} - k^2}\frac{\partial \tilde{B}_{0z}}{\partial y} \quad ,$$

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Waveguide modes: intensity in rectangular waveguides (continued)

Inserting these, and cancelling common factors, we get

$$\langle \boldsymbol{S} \rangle = \frac{1}{8\pi}\left[\frac{i\omega}{\omega^2/c^2 - k^2}\left(-\frac{\partial \tilde{B}_{0z}}{\partial x} B_{0z}^* \hat{\boldsymbol{x}} - \frac{\partial \tilde{B}_{0z}}{\partial y} B_{0z}^* \hat{\boldsymbol{y}}\right)\right.$$

$$\left. + \frac{k\omega}{\left(\omega^2/c^2 - k^2\right)^2}\left(\frac{\partial \tilde{B}_{0z}}{\partial y}\frac{\partial \tilde{B}_{0z}^*}{\partial y} + \frac{\partial \tilde{B}_{0z}}{\partial x}\frac{\partial \tilde{B}_{0z}^*}{\partial x}\right)\hat{\boldsymbol{z}}\right] .$$

We also discovered last time that

$$\tilde{B}_{0z} = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad ,$$

$$m, n = 0, 1, 2, \ldots$$

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Waveguide modes: intensity in rectangular waveguides (continued)

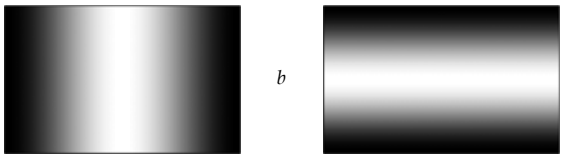
so

And remember, only $\mathrm{Re}\langle \boldsymbol{S} \rangle$ is physical.

$$\langle \boldsymbol{S} \rangle = \frac{B_0^2}{8\pi}\left[\frac{i\omega}{\omega^2/c^2 - k^2}\left(\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\hat{\boldsymbol{x}} + \frac{n\pi}{b}\cos^2\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{n\pi y}{b}\hat{\boldsymbol{y}}\right) + \frac{k\omega}{\left(\omega^2/c^2 - k^2\right)^2}\left(\left[\frac{n\pi}{b}\right]^2\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \left[\frac{m\pi}{b}\right]^2\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\right)\hat{\boldsymbol{z}}\right].$$

Change $\frac{1}{8\pi}$ to $\frac{1}{2\mu_0}$ to get it in MKS.

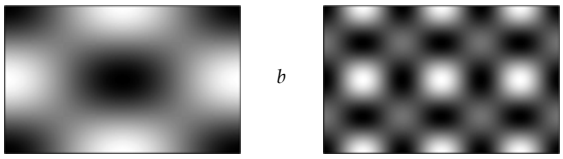
Waveguide modes: intensity in rectangular waveguides (continued)



TE_{10} TE_{01}

These are grayscale plots of the z component of $\boldsymbol{S}$: white is the highest intensity, black is zero intensity, gray is in between.

Waveguide modes: intensity in rectangular waveguides (continued)



TE_{11} TE_{32}

As before: grayscale plots of the z component of $\boldsymbol{S}$: white is the highest intensity, black is zero intensity, gray is in between.

Dispersion and cut-off in waveguides

Clearly, waveguides are dispersive, as indicated by the frequency dependence of the amplitudes of the wave solutions, and of the intensity.

- The wavenumber of the TE solutions comes from the condition we obtained during our separation solution:

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0$$

$$-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + \frac{\omega^2}{c^2} - k^2 = 0 \quad \text{, or}$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}} \quad .$$

Dispersion and cut-off in waveguides (continued)

- Note that if

$$\frac{\omega^2}{c^2} < \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \quad \text{, or}$$

$$\omega < c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \equiv \omega_{mn} \quad ,$$

the wavenumber k is purely imaginary, and the wave attenuates exponentially on its way down the waveguide. Thus there is a **cutoff frequency** for each propagating mode, ω_{mn}, below which the waveguide does not transmit power, and

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \quad .$$

Aside: massive photons

We found for both plasmas and waveguides that there is a lower frequency bound for propagating light, and thus that the relation between wavenumber and frequency is

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad , \quad k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \quad ,$$

or

$$k^2c^2 = \omega^2 - \omega_{\text{cutoff}}^2 \quad .$$

We also know (problem 9.22b) that k is related to momentum and ω to energy, so we can compare this result to the well-known formula from relativistic mechanics,

$$p^2c^2 = E^2 - \left(m_0c^2\right)^2 \quad .$$

Aside: massive photons

- Thus the cutoff frequency of plasmas and waveguides is analogous to rest mass in relativistic particle mechanics.
- This analogy can actually be thought of as the correspondence limit of the mechanics of *photons* in plasmas and waveguides: photons "really do" have mass, if they reside in plasmas or waveguides; the rest mass m_0 would be given by

$$m_0 c^2 = \hbar \omega_{\text{cutoff}} \quad .$$

Dispersion and cut-off in waveguides (continued)

- We have seen before (in plasmas) that dispersion makes the group and phase velocities different. Same here:

$$k = \frac{\omega}{c}\sqrt{1-\omega_{mn}^2/\omega^2} \quad \Rightarrow \quad v = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_{mn}^2/\omega^2}} > c \quad .$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = \left(\frac{d}{d\omega}\frac{\omega}{c}\left(1-\omega_{mn}^2/\omega^2\right)^{1/2} \right)^{-1}$$

$$= \left(\frac{1}{c}\left(1-\omega_{mn}^2/\omega^2\right)^{1/2} + \frac{\omega}{2c}\left(1-\omega_{mn}^2/\omega^2\right)^{-1/2}\left(\frac{-2\omega_{mn}^2}{\omega^3}\right) \right)^{-1}$$

$$= c\sqrt{1-\omega_{mn}^2/\omega^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}+\frac{\omega_{mn}^2}{\omega^2}\right) = c\sqrt{1-\omega_{mn}^2/\omega^2} < c \quad .$$

Boundary conditions and hollow conducting waveguides

Without even referring to the boundary conditions, we already showed that there are no TEM waves in hollow metal waveguides. Here's how the mere fact that boundary conditions *would be applied* gave us this result implicitly.

- We started sought our wave solution for waveguides by hypothesizing plane-wave like fields with amplitudes

$$\tilde{\boldsymbol{E}}_0 = \tilde{E}_{0x}(x,y)\hat{\boldsymbol{x}} + \tilde{E}_{0y}(x,y)\hat{\boldsymbol{y}} + \tilde{E}_{0z}(x,y)\hat{\boldsymbol{z}} \quad ,$$

$$\tilde{\boldsymbol{B}}_0 = \tilde{B}_{0x}(x,y)\hat{\boldsymbol{x}} + \tilde{B}_{0y}(x,y)\hat{\boldsymbol{y}} + \tilde{B}_{0z}(x,y)\hat{\boldsymbol{z}} \quad .$$

- Thus if $\boldsymbol{E}$ has no longitudinal component, Gauss's law says

$$\frac{\partial \tilde{E}_{0x}}{\partial x} + \frac{\partial \tilde{E}_{0y}}{\partial y} = 0 \quad .$$

Boundary conditions and hollow conducting waveguides (continued)

- Furthermore, if $\boldsymbol{B}$ has no longitudinal component, than Faraday's law says

$$\left(\nabla\times\tilde{\boldsymbol{E}}\right)_z = \frac{\partial\tilde{E}_{0y}}{\partial x}e^{i(kz-\omega t)} - \frac{\partial\tilde{E}_{0x}}{\partial y}e^{i(kz-\omega t)} = -\frac{1}{c}\frac{\partial\tilde{B}_z}{\partial t} = 0 \quad \text{, or}$$

$$\frac{\partial\tilde{E}_{0y}}{\partial x} - \frac{\partial\tilde{E}_{0x}}{\partial y} = 0 \quad .$$

- These last two results indicate that the *amplitude* of $\boldsymbol{E}$,

$$\tilde{\boldsymbol{E}}_0 = \tilde{E}_{0x}(x,y)\hat{\boldsymbol{x}} + \tilde{E}_{0y}(x,y)\hat{\boldsymbol{y}} \quad ,$$

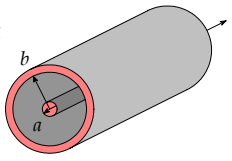
has zero divergence and curl, and thus can be written as the gradient of a scalar potential, $\tilde{\boldsymbol{E}}_0 = -\nabla V_0$.

Boundary conditions and hollow conducting waveguides (continued)

- Now, the vacuum inside the waveguide has no free charges or currents, so this scalar potential must satisfy the Laplace equation, $\nabla^2 V_0 = 0$. One would find the potential by solving this equation subject to the boundary conditions imposed by the waveguide.
- But the waveguide is made of a good conductor, and is therefore an equipotential. So the only solution is a **constant** potential, and therefore no wave. Thus there cannot be TEM waves in hollow conductive pipes.
- If, however, the waveguide is not completely hollow – for instance, if it has a conductor within that isn't in contact with the walls (and can be at a different potential), then this objection vanishes. Such is the case for coax cable…

TEM modes in coaxial waveguide

Consider a hollow cylindrical conductor with another conducting cylinder coaxial with it, as shown. With two disconnected conductors present, there can be nontrivial TEM wave solutions, in which case putting a wave solution into Faraday's law gives:



$$-ik\tilde{E}_{0y}\hat{\boldsymbol{x}} + ik\tilde{E}_{0x}\hat{\boldsymbol{y}} = \frac{i\omega}{c}\left(\tilde{B}_{0x}\hat{\boldsymbol{x}} + \tilde{B}_{0y}\hat{\boldsymbol{y}}\right) \quad ,$$

(compare lecture, 23 February 2004, page 6). Similarly, Ampère's law gives

TEM modes in coaxial waveguide (continued)

$$-ik\tilde{B}_{0y}\hat{\boldsymbol{x}} + ik\tilde{B}_{0x}\hat{\boldsymbol{y}} = -\frac{i\omega}{c}\left(\tilde{E}_{0x}\hat{\boldsymbol{x}} + \tilde{E}_{0y}\hat{\boldsymbol{y}}\right) \quad .$$

Thus we have:

$$k = \frac{\omega}{c} \quad , \quad \tilde{E}_{0x} = \tilde{B}_{0y} \quad , \quad \tilde{E}_{0y} = -\tilde{B}_{0x} \quad , \text{ and:}$$

$$\frac{\partial \tilde{E}_{0x}}{\partial x} + \frac{\partial \tilde{E}_{0y}}{\partial y} = 0 \quad \frac{\partial \tilde{B}_{0x}}{\partial x} + \frac{\partial \tilde{B}_{0y}}{\partial y} = 0$$

$$\frac{\partial \tilde{E}_{0y}}{\partial x} - \frac{\partial \tilde{E}_{0x}}{\partial y} = 0 \quad \frac{\partial \tilde{B}_{0y}}{\partial x} - \frac{\partial \tilde{B}_{0x}}{\partial y} = 0$$

.

TEM modes in coaxial waveguide (continued)

- Thus the *amplitudes* are simply the solutions to the axisymmetric electrostatic and magnetostatic problems! Just stick an $e^{i(kz-\omega t)}$ on, and we'll have the correct wave solution.