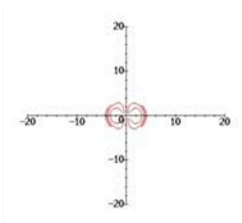
Today in Physics 218: charges, currents, and radiation

- ❑ Retarded potentials and retarded time
- ❑ Retarded potentials and the Lorentz gauge
- ❑ Retarded potentials and the inhomogeneous wave equation



Radiation by two oscillating charges. Animation by Akira Hirose, University of Saskatchewan.

Retarded potentials

- ❑ The electromagnetic waves we've been discussing have to originate somewhere. In the following we'll see that electromagnetic radiation can be generated by
  - time-variable charge and current distributions, and
  - accelerating individual charges.
- ❑ As usual when dealing with charges and currents, it is most convenient to calculate potentials first, and then to obtain fields from the potentials, rather than to calculate the fields directly.
- ❑ Also as usual, we will do our calculations mostly by construction of a solution to the relevant differential equations, demonstration that it works, and reliance upon the uniqueness of solutions.

Retarded potentials (continued)

- ❑ What are the relevant differential equations? As we first saw in lecture on 21 January, we get them from Gauss's and Ampère's laws:

$$\nabla\cdot\boldsymbol{E}=4\pi\rho \quad\Rightarrow\quad \nabla\cdot\left(-\nabla V-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)=4\pi\rho$$

$$\Rightarrow \nabla^2 V+\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{A}=-4\pi\rho \quad ,$$

$$\nabla\times\left(\nabla\times\boldsymbol{A}\right)=\frac{4\pi}{c}\boldsymbol{J}-\frac{1}{c}\frac{\partial}{\partial t}\nabla V-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}$$

$$\nabla\left(\nabla\cdot\boldsymbol{A}\right)-\nabla^2\boldsymbol{A}= \qquad \text{(P.R. \#11)}$$

or $$\left(\nabla^2\boldsymbol{A}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}\right)-\nabla\left(\nabla\cdot\boldsymbol{A}+\frac{1}{c}\frac{\partial V}{\partial t}\right)=-\frac{4\pi}{c}\boldsymbol{J} \quad .$$

Retarded potentials (continued)

- With the Lorentz gauge condition, these equations become

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho \quad , \quad \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{J} \quad ,$$

  that is, inhomogeneous wave equations.
- To construct a solution, first note that we have a lot of experience with the *static* case. For $\partial^2 V \big/ \partial t^2 = 0 = \partial^2 \boldsymbol{A} \big/ \partial t^2$, the potentials obey Poisson equations:

$$\nabla^2 V = -4\pi\rho \quad , \quad \nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J} \quad ,$$

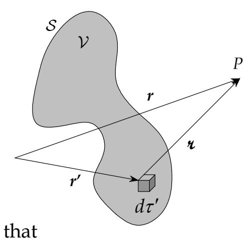
  and in PHY 217 we showed in gory detail that the solutions to these equations are:

Retarded potentials (continued)

$$V(\boldsymbol{r}) = \int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{\mathscr{r}} \quad ,$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}')\,d\tau'}{\mathscr{r}} \quad ,$$

where $\mathcal{V}$ is the volume that contains the charges and currents.



- We've also seen this semester that fields and energy propagate at speed c in vacuum, when they travel in the form of electromagnetic waves.

Retarded potentials (continued)

- Here comes the guess:
  Every infinitesimal element of charge or current is a different distance $\mathscr{r}$ away from us (located at $\boldsymbol{r}$). Thus a change in the sources at time t' and position $\boldsymbol{r}'$ doesn't lead to a change in the fields at $\boldsymbol{r}$ until the **later** time $t' + \mathscr{r}/c$.
- In other words, the fields at $\boldsymbol{r}$ depend upon the condition of the sources at $\boldsymbol{r}'$ at the **earlier** time $t - \mathscr{r}/c$. So we'll guess that

$$V(\boldsymbol{r},t) = \int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}', t - \mathscr{r}/c)\,d\tau'}{\mathscr{r}} \quad , \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c}\int_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r}', t - \mathscr{r}/c)\,d\tau'}{\mathscr{r}} \quad .$$

  These are called the **retarded potentials**.

Retarded potentials (continued)

- $t_r = t - \mathscr{r}/c$ is called the **retarded time** for the positions $\boldsymbol{r}$ and $\boldsymbol{r}'$.
- Now we need to show that these potentials satisfy the Lorentz gauge condition, and are solutions to the inhomogeneous wave equation.
- For the former, we will need to fiddle with the divergence of $\boldsymbol{J}$ for a bit before we're ready to move on to the divergence of $\boldsymbol{A}$. Bear with me for a few slides…

Retarded potentials and the Lorentz gauge

- First, note that the product rule for derivatives means that

$$\nabla\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}\nabla\cdot\boldsymbol{J}+\boldsymbol{J}\cdot\nabla\left(\frac{1}{\mathscr{r}}\right)\quad\text{and}$$

$$\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}\nabla'\cdot\boldsymbol{J}+\boldsymbol{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)\quad,$$

where $\nabla\equiv\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial}{\partial z}$ and $\nabla'\equiv\hat{\boldsymbol{x}}\frac{\partial}{\partial x'}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y'}+\hat{\boldsymbol{z}}\frac{\partial}{\partial z'}$, as usual. Recall also that because $\mathscr{r}=\boldsymbol{r}-\boldsymbol{r}'$,

$$\nabla\left(\frac{1}{\mathscr{r}}\right)=-\nabla'\left(\frac{1}{\mathscr{r}}\right)\quad,$$

as we showed and used frequently in PHY 217.

Retarded potentials and the Lorentz gauge (continued)

- Thus

$$\nabla\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)=\frac{1}{\mathscr{r}}\nabla\cdot\boldsymbol{J}-\boldsymbol{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

$$=\frac{1}{\mathscr{r}}\nabla\cdot\boldsymbol{J}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)+\frac{1}{\mathscr{r}}\nabla'\cdot\boldsymbol{J}\quad.$$

- Now, there's an *implicit* dependence of $\boldsymbol{J}$ on $\boldsymbol{r}$ through $t_r=t-\mathscr{r}/c$ just because $\mathscr{r}=\boldsymbol{r}-\boldsymbol{r}'$. So, using the chain rule,

$$\nabla\cdot\boldsymbol{J}=\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=\frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z}$$

$$=-\frac{1}{c}\left(\frac{\partial J_x}{\partial t_r}\frac{\partial\mathscr{r}}{\partial x}+\frac{\partial J_y}{\partial t_r}\frac{\partial\mathscr{r}}{\partial y}+\frac{\partial J_z}{\partial t_r}\frac{\partial\mathscr{r}}{\partial z}\right)=-\frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t_r}\cdot\nabla\mathscr{r}\quad.$$

Retarded potentials and the Lorentz gauge (continued)

- Without this implicit dependence upon $\boldsymbol{r}$, $\nabla\cdot\boldsymbol{J}$ would be zero, as it was in the static case. Recall that we used to use $\nabla\cdot\boldsymbol{J}=0$ in magnetostatic calculations (*viz.* the Flashback in the lecture notes for 14 January).
- But $\boldsymbol{J}$ depends *explicitly* on $\boldsymbol{r}'$, as well as implicitly through the retarded time t_r, so by the chain rule again,

$$\begin{aligned}\nabla'\cdot\boldsymbol{J} &= \left(\frac{\partial J_{x'}}{\partial x'}+\frac{\partial J_{x'}}{\partial t_r}\frac{\partial t_r}{\partial x'}\right)+\left(\frac{\partial J_{y'}}{\partial y'}+\frac{\partial J_{y'}}{\partial t_r}\frac{\partial t_r}{\partial y'}\right)+\left(\frac{\partial J_{z'}}{\partial z'}+\frac{\partial J_{z'}}{\partial t_r}\frac{\partial t_r}{\partial z'}\right)\\ &= \left[\frac{\partial J_{x'}}{\partial x'}+\frac{\partial J_{y'}}{\partial y'}+\frac{\partial J_{z'}}{\partial z'}\right]-\frac{1}{c}\left[\frac{\partial J_{x'}}{\partial t_r}\frac{\partial \mathscr{r}}{\partial x'}+\frac{\partial J_{y'}}{\partial t_r}\frac{\partial \mathscr{r}}{\partial y'}+\frac{\partial J_{z'}}{\partial t_r}\frac{\partial \mathscr{r}}{\partial z'}\right]\\ &= -\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t}-\frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t_r}\cdot\nabla'\mathscr{r}\quad .\end{aligned}$$

Retarded potentials and the Lorentz gauge (continued)

Note that the continuity equation, $\nabla\cdot\boldsymbol{J}+\partial\rho/\partial t=0$, was used in the last step.

- Combine these last three results:

$$\begin{aligned}\nabla\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right) &= \frac{1}{\mathscr{r}}\nabla\cdot\boldsymbol{J}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)+\frac{1}{\mathscr{r}}\nabla'\cdot\boldsymbol{J} \qquad \text{because } \nabla\mathscr{r}=-\nabla'\mathscr{r}\\ &= \frac{1}{\mathscr{r}}\left(\cancel{-\frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t_r}\cdot\nabla\mathscr{r}}\right)-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)+\frac{1}{\mathscr{r}}\left(-\frac{\partial\rho}{\partial t}-\cancel{\frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t_r}\cdot\nabla'\mathscr{r}}\right)\\ &= -\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)\quad .\end{aligned}$$

- We can use this in the form of $\boldsymbol{A}$ we've guessed, and verify obedience to the Lorentz gauge condition:

Retarded potentials and the Lorentz gauge (continued)

$$\begin{aligned}\nabla\cdot\boldsymbol{A} &= \nabla\cdot\frac{1}{c}\int_{\mathcal{V}}\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)\,d\tau'}{\mathscr{r}}=\frac{1}{c}\int_{\mathcal{V}}\nabla\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)d\tau'\\ &= \frac{1}{c}\int_{\mathcal{V}}\left(-\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t}-\nabla'\cdot\left(\frac{\boldsymbol{J}}{\mathscr{r}}\right)\right)d\tau' \qquad \text{Use the divergence theorem:}\\ &= -\frac{1}{c}\frac{\partial}{\partial t}\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r}',t_r)}{\mathscr{r}}d\tau'-\frac{1}{c}\oint_{\mathcal{S}}\frac{\boldsymbol{J}\cdot d\boldsymbol{a}'}{\mathscr{r}}=-\frac{1}{c}\frac{\partial V}{\partial t}-\frac{1}{c}\oint_{\mathcal{S}}\frac{\boldsymbol{J}\cdot d\boldsymbol{a}'}{\mathscr{r}}\quad .\end{aligned}$$

- The last term vanishes if we choose the surface $\mathcal{S}$ to enclose **all** of the charges and currents, because no current flows through that surface, by definition (so $\boldsymbol{J}$ = 0 there):

$$\nabla\cdot\boldsymbol{A}=-\frac{1}{c}\frac{\partial V}{\partial t}\quad . \qquad \text{Lorentz gauge}$$

The solutions to the inhomogeneous wave equations are retarded potentials

- Now we are in a position to see whether the retarded potentials are solutions to the wave equations we derived from the Maxwell equations.
- Start by computing the Laplacian of V, and aim at showing that this is equal to $\dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}+4\pi\rho$. First, we'll need to fiddle with the gradient of V a bit:

$$\nabla V(\boldsymbol{r},t)=\nabla\int_{\mathcal{V}}\frac{\rho(\boldsymbol{r}',t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}}=\int_{\mathcal{V}}\left[\frac{\nabla\rho}{\boldsymbol{\mathscr{r}}}+\rho\nabla\left(\frac{1}{\boldsymbol{\mathscr{r}}}\right)\right]d\tau' \quad .$$

- $\rho(r',t_r)$ depends implicitly on $\boldsymbol{r}$, through $t_r=t-\boldsymbol{\mathscr{r}}/c$, so

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The solutions to the inhomogeneous wave equations are retarded potentials (continued)

$$\nabla\rho=\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial x}\hat{x}+\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial y}\hat{y}+\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial z}\hat{z}=\frac{\partial\rho}{\partial t_r}\nabla t_r$$
$$=-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\nabla\boldsymbol{\mathscr{r}} \quad .$$

But, as we showed in PHY 217,

$$\nabla\boldsymbol{\mathscr{r}}=\hat{\boldsymbol{\mathscr{r}}} \quad \text{, and} \quad \nabla\left(\frac{1}{\boldsymbol{\mathscr{r}}}\right)=-\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \quad ,$$

so

$$\nabla V(\boldsymbol{r},t)=\int_{\mathcal{V}}\left[-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}}-\rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\right]d\tau' \quad .$$

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