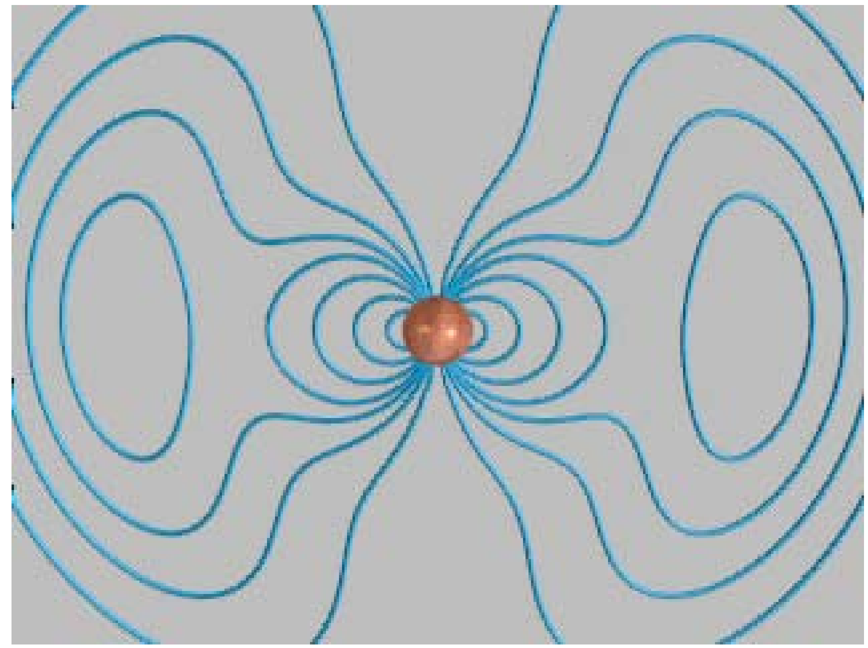
Today in Physics 218: electric dipole radiation

- ❑ Retarded potentials as solutions to the inhomogeneous wave equation
- ❑ Example: retarded potentials for an oscillating electric dipole



Radiation by sinusoidally-varying magnetic dipole. Animation by John Belcher, MIT.

The solutions to the inhomogeneous wave equations are retarded potentials

- Last time we began to compute the Laplacian of V, and thus to aim at showing that this is equal to $\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + 4\pi\rho$.

  We got as far as

$$\nabla V(\boldsymbol{r},t) = \int_{\mathcal{V}} \left[-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\frac{\hat{\boldsymbol{\imath}}}{\imath} - \frac{1}{c}\rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right] d\tau' \quad .$$

- Now we have to take the divergence of this result:

$$\nabla^2 V = \nabla\cdot(\nabla V) = -\int_{\mathcal{V}} \nabla\cdot\left[\frac{1}{c}\frac{\partial\rho}{\partial t_r}\frac{\hat{\boldsymbol{\imath}}}{\imath} + \rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right] d\tau' \quad .$$

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

- Use the chain rule:

$$\nabla^2 V = \nabla \cdot (\nabla V)$$

$$= -\int_{\mathcal{V}} \left[\frac{1}{c} \nabla \left(\frac{\partial \rho}{\partial t_r} \right) \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} + \frac{1}{c} \frac{\partial \rho}{\partial t_r} \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} + (\nabla \rho) \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} + \rho \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right] d\tau' \quad .$$

- We already showed and used, last time, the fact that

$$\nabla \rho = -\frac{1}{c} \frac{\partial \rho}{\partial t_r} \nabla \mathscr{r} = -\frac{1}{c} \frac{\partial \rho}{\partial t_r} \hat{\boldsymbol{\mathscr{r}}} \quad ,$$

so, using the chain rule again,

$$\nabla \frac{\partial \rho}{\partial t_r} = \frac{\partial}{\partial t_r} \left(\frac{\partial \rho}{\partial t_r} \right) \nabla t_r = -\frac{1}{c} \frac{\partial^2 \rho}{\partial t_r^2} \nabla \mathscr{r} = -\frac{1}{c} \frac{\partial^2 \rho}{\partial t_r^2} \hat{\boldsymbol{\mathscr{r}}} \quad .$$

Flashback: concerning $\nabla\cdot\left(\hat{\boldsymbol{r}}/r^2\right)$.

$$\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\hat{\boldsymbol{r}}}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\hat{\boldsymbol{r}}\right)=0,$$

except at the origin, where it's undefined. On the other hand,

$$\int_{\mathcal{V}}\nabla\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)d\tau=\oint_{\mathcal{S}}\frac{\hat{\boldsymbol{r}}}{r^2}\cdot d\boldsymbol{a}=\oint_{\mathcal{S}}\sin\theta d\theta d\phi=4\pi \quad .$$

This should remind you of the behaviour of the delta function:

$$\int_{\mathcal{V}}\delta^3\left(\boldsymbol{r}\right)d\tau=1 \quad .$$

Thus

$$\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r^2}=4\pi\delta^3\left(\boldsymbol{r}\right) \quad .$$

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

- Also, we proved in PHY 217 that

$$\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}} = \frac{1}{\mathscr{r}^2} \quad \text{and} \quad \nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\mathscr{r}) \quad .$$

- If we put these last four equations into the integral, we get

$$\nabla^2 V = -\int_{\mathcal{V}} \left[-\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t_r^2}\hat{\mathscr{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}} + \cancel{\frac{1}{c}\frac{\partial \rho}{\partial t_r}\frac{1}{\mathscr{r}^2}} - \cancel{\frac{1}{c}\frac{\partial \rho}{\partial t_r}\hat{\mathscr{r}} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} + 4\pi\rho\delta^3(\mathscr{r}) \right] d\tau'$$

$$= \frac{1}{c^2}\int_{\mathcal{V}} \frac{\partial^2 \rho}{\partial t_r^2}\frac{1}{\mathscr{r}} d\tau' - 4\pi\rho(\boldsymbol{r}, t) \quad .$$

- Finally, note that

$$\frac{\partial}{\partial t} = \frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t_r} \quad \Rightarrow \quad \frac{\partial^2 \rho}{\partial t_r^2}\frac{1}{\mathscr{r}} = \frac{\partial^2}{\partial t^2}\left(\frac{\rho}{\mathscr{r}}\right) \quad ,$$

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

- …and we have what we were looking for:

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}', t_r)}{\boldsymbol{\mathscr{r}}}\,d\tau' - 4\pi\rho(\boldsymbol{r}, t)$$

$$= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - 4\pi\rho(\boldsymbol{r}, t) \quad ;$$

that is, the retarded potential

$$V(\boldsymbol{r}, t) = \int_{\mathcal{V}} \frac{\rho(\boldsymbol{r}', t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}}$$

is a solution to the inhomogeneous wave equation.

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

- This process can be replicated for each component of $\boldsymbol{A}$ (and $\boldsymbol{J}$) to show that the retarded vector potential is a solution to the inhomogenous wave equation, too.
- Griffiths refers also to an indirect proof that the retarded potentials are solutions the inhomogeneous wave equations. That's the proof I learned as an undergraduate. It's actually not very indirect, and since it was actually first done by Riemann (1858) even before Maxwell completed his equations (early to mid 1860s), and since it counts as the first demonstration that electromagnetic waves travel at the speed of light (Riemann claimed this "established the connection between electricity and optics"), it may be worth including here as a curiosity.

Curiosity: Riemann's proof

Subdivide $\mathcal{V}$ into two parts: a very small volume ($\mathcal{V}_1$) centered on $\boldsymbol{r}$, and the rest ($\mathcal{V}_2$). The potential is the superposition of contributions from these two volumes:

$$V(\boldsymbol{r},t) = V_1 + V_2 = \int_{\mathcal{V}_1} \frac{\rho(\boldsymbol{r}',t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}} + \int_{\mathcal{V}_2} \frac{\rho(\boldsymbol{r}',t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}} \quad ,$$

where V is the electric potential, or c times any component of the vector potential, and ρ is the charge density, or the corresponding component of current density. If $\mathcal{V}_1$ is sufficiently small, retardation effects are small, so

$$V_1 = \frac{1}{c}\int_{\mathcal{V}_1} \frac{\rho(\boldsymbol{r}',t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}} \rightarrow \frac{1}{c}\int_{\mathcal{V}_1} \frac{\rho(\boldsymbol{r}',t)\,d\tau'}{\boldsymbol{\mathscr{r}}} \quad ,$$

Curiosity: Riemann's proof (continued)

which is the electrostatic solution, so

$$\nabla^2 V_1(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t) \quad .$$

Consider the viewpoint of an infinitesimal volume element within $\mathcal{V}_2$, $d\tau'$. The charge density appears *spherically symmetrical* in $\mathscr{r}$, because $\mathscr{r}$ is linear in r. Thus

$$\nabla^2\left(\frac{\rho}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}\left(\mathscr{r}^2\,\frac{\partial}{\partial \mathscr{r}}\frac{\rho}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}\left(\mathscr{r}^2\left(\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial \mathscr{r}} - \frac{\rho}{\mathscr{r}^2}\right)\right)$$

$$= \frac{1}{\mathscr{r}^2}\left(\frac{\partial\rho}{\partial \mathscr{r}} + \mathscr{r}\frac{\partial^2\rho}{\partial \mathscr{r}^2} - \frac{\partial\rho}{\partial \mathscr{r}}\right) = \frac{1}{\mathscr{r}}\frac{\partial^2\rho}{\partial \mathscr{r}^2} \quad .$$

Curiosity: Riemann's proof (continued)

Thus,

$$\nabla^2 V_2(\mathbf{r},t) = \int_{\mathcal{V}_2} \nabla^2 \frac{\rho(\mathbf{r}',t_r)\,d\tau'}{\boldsymbol{\mathscr{r}}} = \int_{\mathcal{V}_2} \frac{1}{\boldsymbol{\mathscr{r}}} \frac{\partial^2 \rho(\mathbf{r}',t_r)}{\partial \boldsymbol{\mathscr{r}}^2} d\tau'$$

Now note one of our results from early this semester (lecture, 26 January): any function of $t_{\boldsymbol{\mathscr{r}}} = t - \boldsymbol{\mathscr{r}}/c$ is a solution to the one-dimensional wave equation, with speed c. (!!) So,

$$\frac{\partial^2 \rho}{\partial \boldsymbol{\mathscr{r}}^2} = \frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} \quad \Rightarrow$$

$$\nabla^2 V_2(\mathbf{r},t) = \frac{1}{c^2} \int_{\mathcal{V}_2} \frac{1}{\boldsymbol{\mathscr{r}}} \frac{\partial^2 \rho(\mathbf{r}',t_r)}{\partial t^2} d\tau' = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathcal{V}_2} \frac{\rho(\mathbf{r}',t_r)}{\boldsymbol{\mathscr{r}}} d\tau' = \frac{1}{c^2} \frac{\partial^2 V_2}{\partial t^2}.$$

Curiosity: Riemann's proof (continued)

And we can now let $\mathcal{V}_1 \to 0, \mathcal{V}_2 \to \mathcal{V}$:

$$\nabla^2 \left(V_1 + V_2\right) = -4\pi\rho\left(\boldsymbol{r}, t\right) + \frac{1}{c^2}\frac{\partial^2 V_2}{\partial t^2}$$

$$\rightarrow \quad \nabla^2 V = -4\pi\rho\left(\boldsymbol{r}, t\right) + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \quad .$$

an inhomogeneous wave equation, as advertised.

The solutions to the inhomogeneous wave equations are retarded potentials (continued)

- The same reasoning as above actually applies also to the **advanced** potentials,

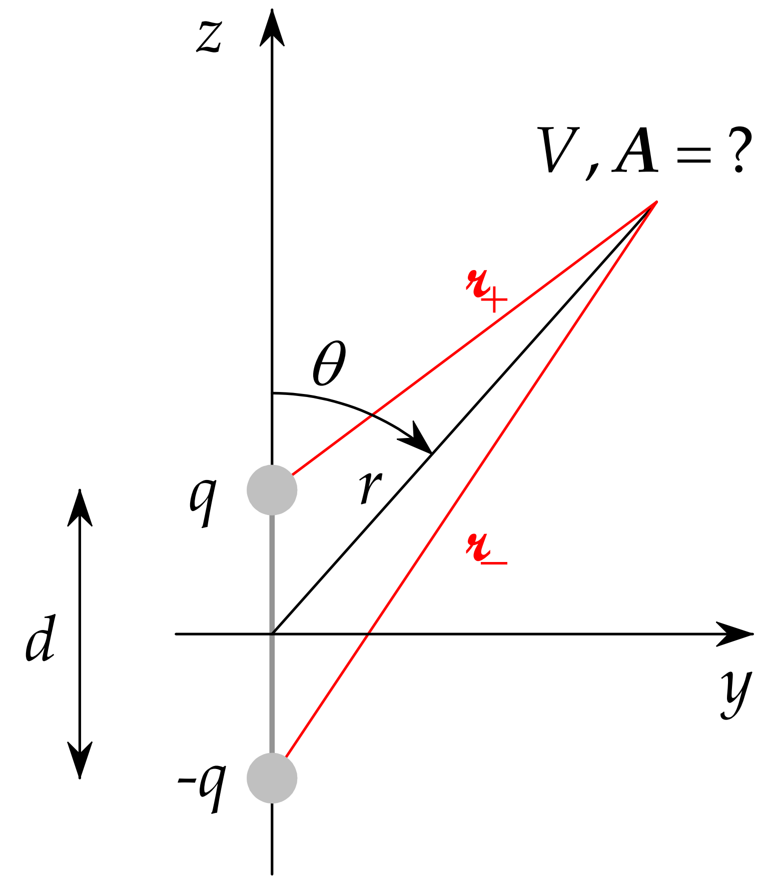
$$V_a(\boldsymbol{r},t) = \int_{\mathcal{V}} \frac{\rho\left(\boldsymbol{r}',t+\boldsymbol{\imath}/c\right)d\tau'}{\boldsymbol{\imath}} \quad , \quad \boldsymbol{A}_a(\boldsymbol{r},t) = \frac{1}{c}\int_{\mathcal{V}} \frac{\boldsymbol{J}\left(\boldsymbol{r}',t+\boldsymbol{\imath}/c\right)d\tau'}{\boldsymbol{\imath}} \quad ,$$

as might be expected, since – like most of the equations of physics – the equations of electrodynamics are time-reversal invariant. So *they* are solutions to the inhomogeneous wave equation too. The advanced potentials lack physical significance, though, because they violate causality.

Electric dipole radiation

The simplest example of a radiating system is perhaps the oscillating electric dipole.

- ❑ Consider a dipole consisting of two conducting spheres carrying charge $q = \pm q_0 \cos \omega t$, separated by a thin wire of length d, and arranged along the z axis as shown. What are the scalar and vector potentials at some distance $r \gg d$?



Electric dipole radiation (continued)

Solution:

- The distance, and light propagation times, from the two charge to us (sitting at *r*), are different. By the law of cosines,

$$\mathscr{r}_{\pm} = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 \mp 2r\frac{d}{2}\cos\theta} \quad ,$$

  since $\cos(\pi - \theta) = -\cos\theta$. If $r \gg d$, we can approximate:

$$\mathscr{r}_{\pm} = r\sqrt{1 \mp \frac{d}{r}\cos\theta + \cancel{\left(\frac{d}{2r}\right)^2}} \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right) \quad ;$$

$$\frac{1}{\mathscr{r}_{\pm}} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad .$$

Electric dipole radiation (continued)

- With this formula we can write the electric potential as the sum of the retarded potentials of each charge:

$$V = \frac{q_0 \cos\omega\left(t - \mathscr{r}_+/c\right)}{\mathscr{r}_+} - \frac{q_0 \cos\omega\left(t - \mathscr{r}_-/c\right)}{\mathscr{r}_-}$$

$$= \frac{q_0}{r}\left(1 + \frac{d\cos\theta}{2r}\right)\cos\omega\left(t - \frac{r}{c} - \frac{d\cos\theta}{2c}\right)$$

$$- \frac{q_0}{r}\left(1 - \frac{d\cos\theta}{2r}\right)\cos\omega\left(t - \frac{r}{c} + \frac{d\cos\theta}{2c}\right) \quad .$$

- The sinusoidally-varying factors can be simplified by using $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$:

Electric dipole radiation (continued)

$$\cos\left[\omega\left(t-\frac{r}{c}\right)\pm\frac{\omega d}{2c}\cos\theta\right]=\cos\omega\left(t-\frac{r}{c}\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right)$$
$$\mp\sin\omega\left(t-\frac{r}{c}\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right)\quad .$$

- At this point we need to specify more about the oscillation frequency or wavelength. Let's suppose that the dipole is much smaller than any wavelength of interest, as well as being much smaller than our distance from it:
$$d \ll c/\omega = \lambda/2\pi \quad .$$
Then, to first order in the small-angle approximation,
$$\cos\left(\frac{\omega d}{2c}\cos\theta\right)\cong 1 \quad , \quad \sin\left(\frac{\omega d}{2c}\cos\theta\right)\cong\frac{\omega d}{2c}\cos\theta \quad :$$

Electric dipole radiation (continued)

$$\cos\left[\omega\left(t-\frac{r}{c}\right)\pm\frac{\omega d}{2c}\cos\theta\right]\cong\cos\omega\left(t-\frac{r}{c}\right)\mp\frac{\omega d}{2c}\cos\theta\sin\omega\left(t-\frac{r}{c}\right)\quad ,$$

and the potential at our measurement point is

$$V=\frac{q_0}{r}\left(1+\frac{d\cos\theta}{2r}\right)\left[\cos\omega\left(t-\frac{r}{c}\right)-\frac{\omega d}{2c}\cos\theta\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$-\frac{q_0}{r}\left(1-\frac{d\cos\theta}{2r}\right)\left[\cos\omega\left(t-\frac{r}{c}\right)+\frac{\omega d}{2c}\cos\theta\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$=2\frac{q_0 d\cos\theta}{2r^2}\cos\omega\left(t-\frac{r}{c}\right)-2\frac{q_0 d\omega\cos\theta}{2rc}\sin\omega\left(t-\frac{r}{c}\right)$$

$$=2\frac{p_0\cos\theta}{2r^2}\cos\omega\left(t-\frac{r}{c}\right)-2\frac{p_0\omega\cos\theta}{2rc}\sin\omega\left(t-\frac{r}{c}\right)\quad .$$