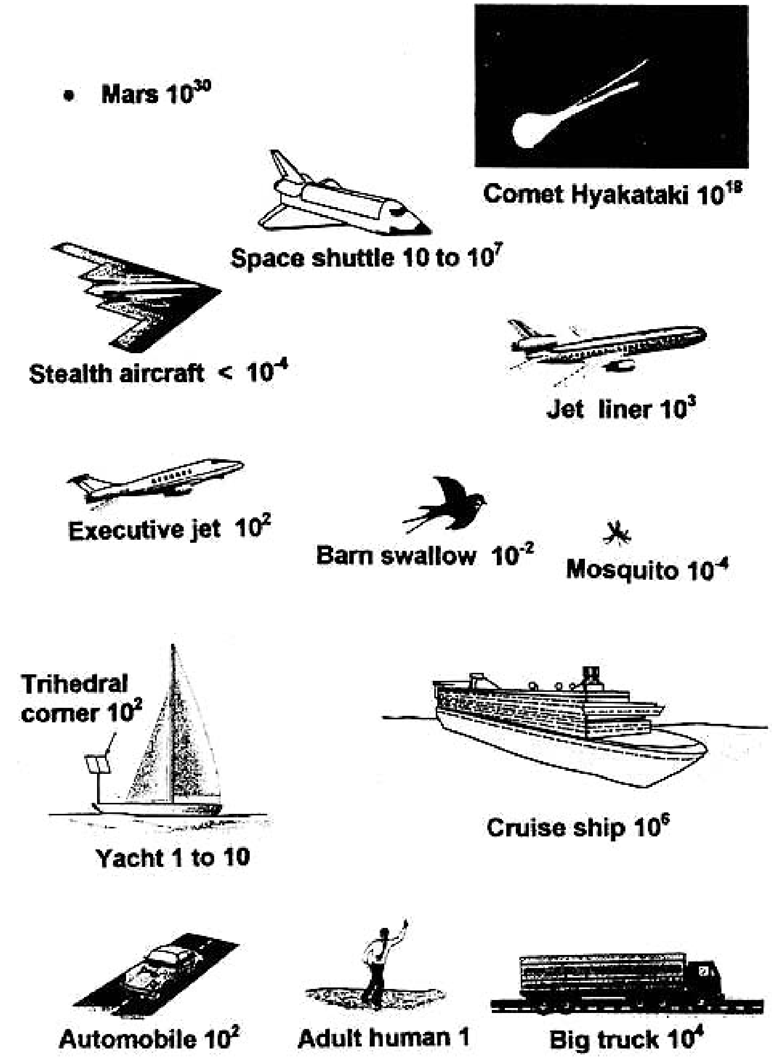
Today in Physics 218: electric dipole radiation II

- The far field
- Vector potential for an oscillating electric dipole
- Radiated fields and intensity for an oscillating electric dipole
- Total scattering cross section of a dielectric sphere



Radar scattering cross section, in square meters at frequency 10 GHz, of some objects ordinarily observed by radar systems. From Antennas, *by J.D. Kraus and R. Marhefka.*

Far-field radiation from an oscillating electric dipole

Last time we found that the scalar potential for an oscillating electric dipole is

$$V = \frac{p_0 \cos\theta}{r^2} \cos\omega\left(t - \frac{r}{c}\right) - \frac{p_0 \omega \cos\theta}{rc} \sin\omega\left(t - \frac{r}{c}\right) \quad ,$$

where $p_0 = q_0 d$, and q_0 is the amplitude of the two separated electric charges. There are at least two interesting limits in which to view this solution:

- DC ($\omega = 0$). Then we get a familiar result:

  $$V_{DC} = \frac{p_0 \cos\theta}{r^2} \quad ,$$

  the same as we got in PHY 217 for a static electric dipole. (http://www.pas.rochester.edu/~dmw/phy217/Lectures/Lect_20b.pdf).

Far-field radiation from an oscillating electric dipole (continued)

- High frequencies and large r. In this case,

$$r \gg c/\omega = \lambda/2\pi \quad ,$$

Far field (or "radiation zone")

so that all told we have assumed $r \gg \lambda/2\pi \gg d$. That makes the second term in V much bigger than the first:

$$V_{\text{rad}}(r,\theta,t) = -\frac{p_0\omega\cos\theta}{rc}\sin\omega\left(t-\frac{r}{c}\right) \quad .$$

Meanwhile, the current in the wire that connects the two charged conducting balls is

$$I = \frac{dq}{dt} = -q_0\omega\sin\omega t \quad ,$$

Vector potential for an oscillating electric dipole

so the retarded vector potential, in the far field, is

$$\boldsymbol{A}_{\mathrm{rad}} = \frac{1}{c}\int \frac{I d\ell}{\boldsymbol{\mathscr{r}}} = \frac{1}{c}\int_{-d/2}^{d/2} -\frac{q_0 \omega \sin \omega\left(t - \boldsymbol{\mathscr{r}}/c\right)\hat{\mathbf{z}} dz'}{\boldsymbol{\mathscr{r}}} \quad .$$

The range of integration isn't much compared to the distances involved in the integrand. Consider, therefore,

$$\int_{-a}^{a} F(x)\, dx = f(a) - f(-a) \quad \text{for small } a \ (\ll 1):$$

$$= \left[f(0) + \left.\frac{df}{dx}\right|_{x=0} a + \ldots \right] - \left[f(0) + \left.\frac{df}{dx}\right|_{x=0} (-a) + \ldots \right]$$

$$\cong 2a\left(\left.\frac{df}{dx}\right|_{x=0} \right) = 2aF(0) \quad .$$

Note that $z = 0$ means $\boldsymbol{\mathscr{r}} = r$.

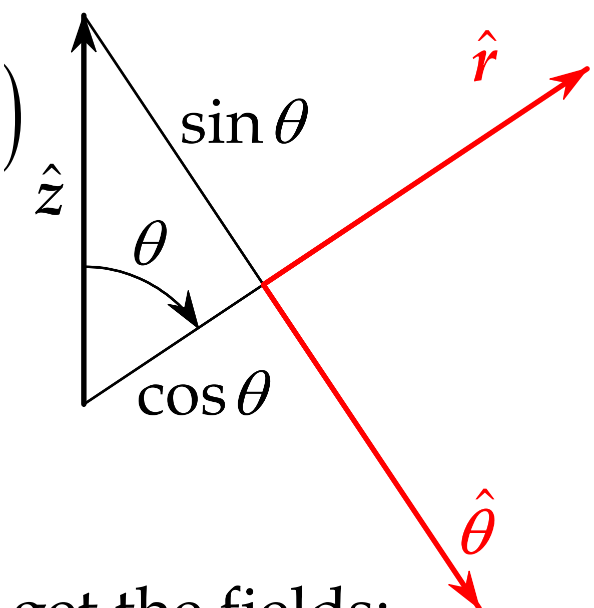
Vector potential for an oscillating electric dipole (continued)

Thus

$$\boldsymbol{A}_{\text{rad}} = -d\frac{q_0\omega\sin\omega\left(t-r/c\right)\hat{\boldsymbol{z}}}{rc} = -\frac{p_0\omega\sin\omega\left(t-r/c\right)}{rc}\hat{\boldsymbol{z}} \quad .$$

Let's convert this to spherical coordinates, to match the expression for V:

$$\boldsymbol{A}_{\text{rad}} = -\frac{p_0\omega\sin\omega\left(t-r/c\right)}{rc}\left(-\hat{\boldsymbol{\theta}}\sin\theta+\hat{\boldsymbol{r}}\cos\theta\right)$$
$$= \frac{p_0\omega\sin\theta\sin\omega\left(t-r/c\right)}{rc}\hat{\boldsymbol{\theta}}$$
$$-\frac{p_0\omega\cos\theta\sin\omega\left(t-r/c\right)}{rc}\hat{\boldsymbol{r}} \quad .$$



From here it's a straightforward matter to get the fields:

Radiated fields and intensity for an oscillating electric dipole

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

$$= \left[\frac{p_0 \omega \cos\theta}{r^2 c}\sin\omega\left(t - \frac{r}{c}\right) - \frac{p_0 \omega^2 \cos\theta}{r c^2}\cos\omega\left(t - \frac{r}{c}\right)\right]\hat{\boldsymbol{r}}$$

$$+ \left[\frac{p_0 \omega \sin\theta}{r^2 c}\sin\omega\left(t - \frac{r}{c}\right)\right]\hat{\boldsymbol{\theta}}$$

$$+ \left[-\frac{p_0 \omega^2 \sin\theta}{r c^2}\cos\omega\left(t - \frac{r}{c}\right)\hat{\boldsymbol{\theta}} + \frac{p_0 \omega^2 \cos\theta}{r c^2}\cos\omega\left(t - \frac{r}{c}\right)\hat{\boldsymbol{r}}\right]$$

$$\cong \boxed{-\frac{p_0 \omega^2 \sin\theta}{r c^2}\cos\omega\left(t - \frac{r}{c}\right)\hat{\boldsymbol{\theta}}} \quad .$$

Radiated fields and intensity for an oscillating electric dipole (continued)

Similarly,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi}$$

$$= \frac{p_0\omega}{rc}\left[\sin\theta\frac{\partial}{\partial r}\left(\sin\omega\left(t - r/c\right)\right) - \frac{\sin\omega\left(t - r/c\right)}{r}\frac{\partial}{\partial\theta}\cos\theta\right]\hat{\phi}$$

$$= \frac{p_0\omega}{rc}\left[-\frac{\omega}{c}\sin\theta\cos\omega\left(t - r/c\right) + \frac{\sin\theta\sin\omega\left(t - r/c\right)}{r}\right]\hat{\phi}$$

$$\cong \frac{p_0\omega}{rc}\left[-\frac{\omega}{c}\sin\theta\cos\omega\left(t - r/c\right)\right]\hat{\phi} = \boxed{-\hat{\phi}\,\frac{p_0\omega^2}{c^2}\frac{\sin\theta}{r}\cos\omega\left(t - r/c\right)} \quad .$$

Radiated fields and intensity for an oscillating electric dipole (continued)

Since $\boldsymbol{k} = \omega/c$, $\boldsymbol{p}_0 = p_0\hat{\boldsymbol{z}}$, and $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{\phi}}$, we can write a more compact equivalent to these formulas:

$$\boxed{\begin{aligned} \boldsymbol{B} &= k^2 \left(\hat{\boldsymbol{r}} \times \boldsymbol{p}_0\right) \frac{\sin\theta}{r} \cos\omega\left(t - \frac{r}{c}\right) \quad , \\ \boldsymbol{E} &= \boldsymbol{B} \times \hat{\boldsymbol{r}} \quad . \end{aligned}}$$

Both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\hat{\boldsymbol{r}}$. Since the $\cos\omega\left(t - r/c\right)$ factor makes this a wave that travels toward $+r$, the far-field radiation from an oscillating electric dipole is a transverse, expanding, **spherical** wave.

Radiated fields and intensity for an oscillating electric dipole (continued)

The Poynting vector for these radiated fields comes out rather simply:

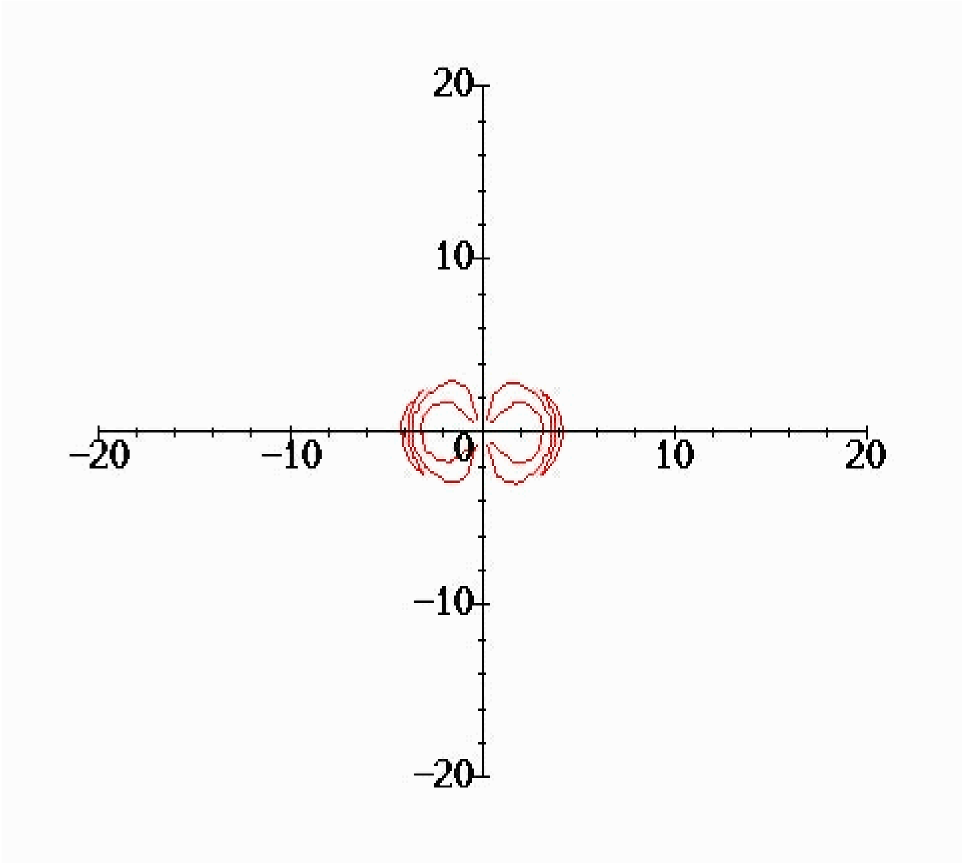
$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}\left(\frac{p_0\omega^2}{c^2}\frac{\sin\theta}{r}\right)^2 \cos^2\omega\left(t - r/c\right)\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} \quad .$$

Noting that $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}$, and recalling that $\left\langle\cos^2\left(\omega t - \delta\right)\right\rangle = 1/2$, this gives us

$$\left\langle\boldsymbol{S}\right\rangle = \frac{c}{8\pi}\left(\frac{p_0\omega^2}{c^2}\right)^2\left(\frac{\sin\theta}{r}\right)^2\hat{\boldsymbol{r}} = I\hat{\boldsymbol{r}} \quad .$$

Radiated fields and intensity for an oscillating electric dipole (continued)

Contour map of the magnitude of the Poynting vector as a function of time, for an oscillating electric dipole. Animation by Akira Hirose, U. Saskatchewan.



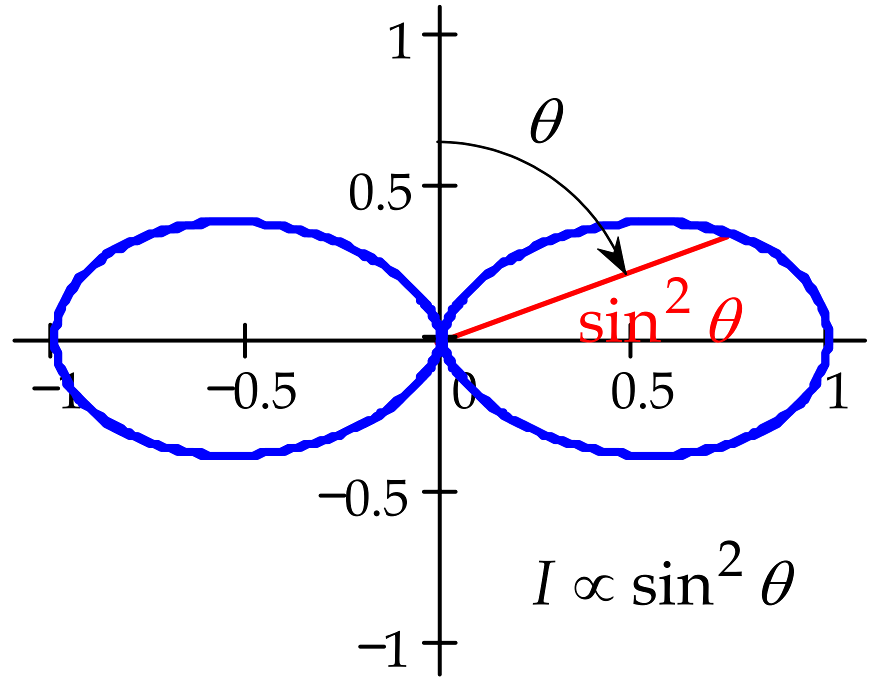
Prof. Hirose's animations can be found at:

http://physics.usask.ca/~hirose/ep225/radiation.htm

Radiated fields and intensity for an oscillating electric dipole (continued)

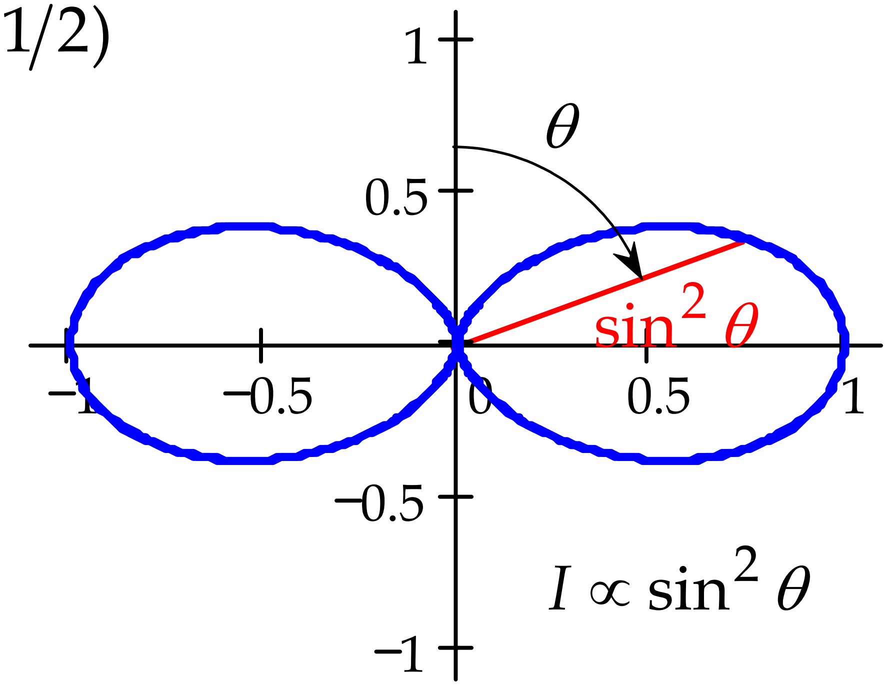
A convenient way to envision the 3-D distribution of this radiated intensity is to plot, in polar coordinates, the intensity for a given distance *r*.

- Because of the $\sin^2\theta$ factor, the intensity is largest for $\hat{\boldsymbol{r}}$ pointing toward $\theta = \pi/2$, and is zero along the z axis. In between it looks like the plot on the right.



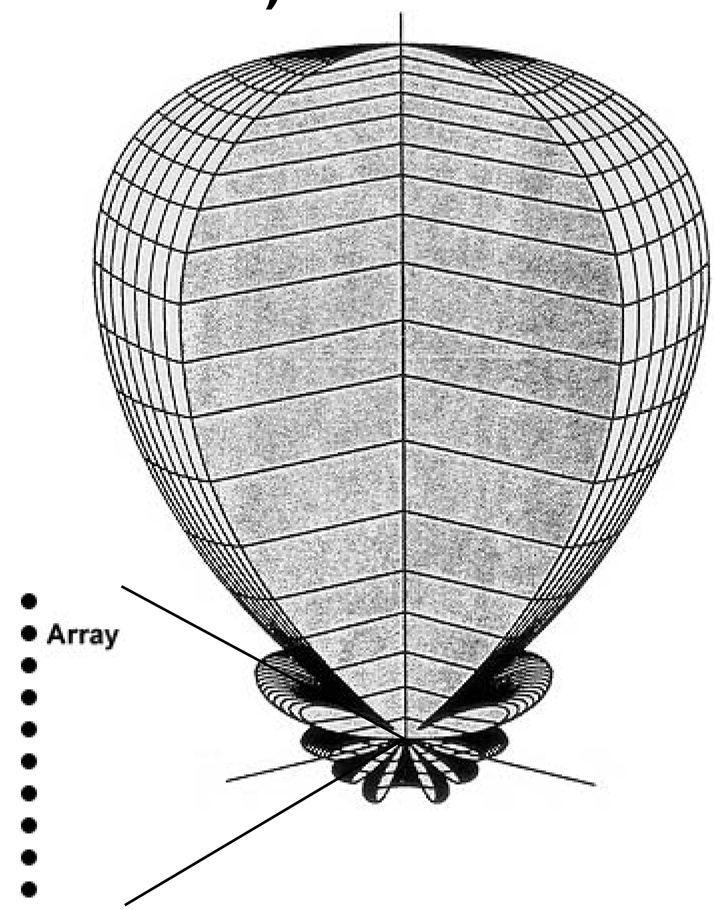
Radiated fields and intensity for an oscillating electric dipole (continued)

- For example: the intensity travelling at $\theta = 30°$ $(\sin\theta = 1/2)$ is half the intensity at $\theta = 45°$ $(\sin\theta = 1/\sqrt{2})$, though in each case the direction is radially outward, and the fields are still transverse (perpendicular to $\hat{\boldsymbol{r}}$).



Radiated fields and intensity for an oscillating electric dipole (continued)

- These kinds of polar plots, that show which way the radiated intensity is greatest, and by how much, are often used to describe the directivity of antennas more complicated than the dipole.
- Here, for example, is what you'd get for ten dipoles in a row, spaced $\lambda/4$ apart with phase shifts of $\pi/2$ between consecutive currents (from Kraus, *Antennas*, 3rd edition).



Total scattering cross section of a dielectric sphere

The total power emitted by a dipole is computed by integrating the Poynting vector over any surface that encloses the dipole, like any sphere for which the radius puts it in the far field:

$$\langle P \rangle = \int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = \frac{p_0^2 \omega^4}{8\pi c^3} \int \frac{\sin^2\theta}{r^2} \hat{\boldsymbol{r}} \cdot r^2 \sin\theta d\theta d\phi \hat{\boldsymbol{r}}$$

$$= \frac{p_0^2 \omega^4}{8\pi c^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta = \frac{p_0^2 \omega^4}{8\pi c^3} 2\pi \frac{4}{3} = \frac{p_0^2 \omega^4}{3c^3} \quad .$$

It doesn’t matter how this dipole is made to oscillate; once it does, though, it will radiate as described, with the total power given above.

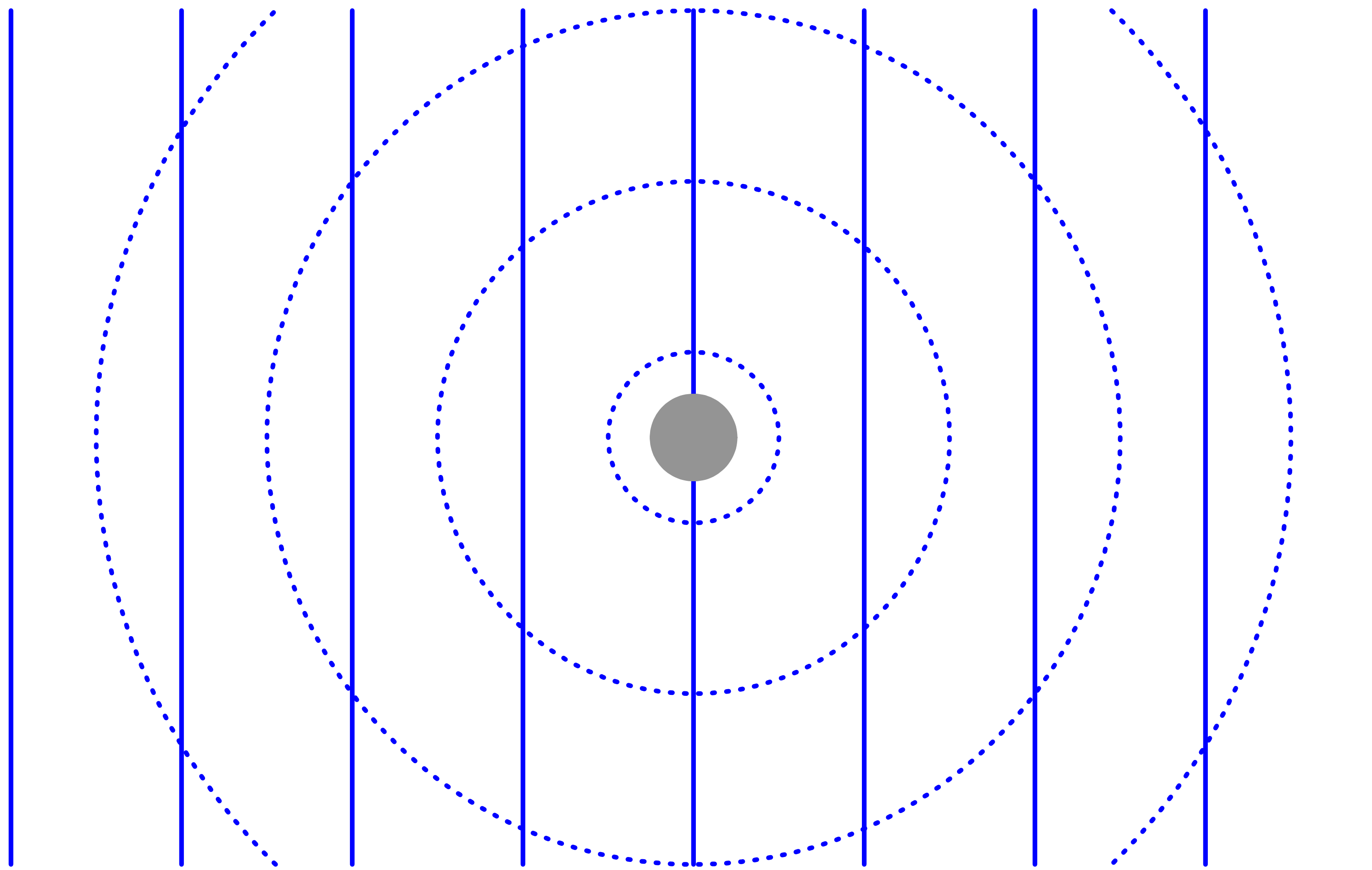
Total scattering cross section of a dielectric sphere (continued)

Suppose, instead of the dipole we started with, we have a dipole *induced* by an oscillating external electric field $\mathbf{E}_0 \cos\omega t$ – for example, a small dielectric sphere, with radius a small enough to consider that there is no delay in propagation of the field across it. The dipole moment of such a sphere is

$$\mathbf{p} = \frac{4\pi}{3} a^3 \mathbf{P} = \frac{4\pi}{3} a^3 \chi_e \mathbf{E}_0 \cos\omega t \equiv \mathbf{p}_0 \cos\omega t \quad .$$

Suppose furthermore that the external electric field is supplied by a plane electromagnetic wave that's passing by the cube. The power radiated by the dipole is thus traceable back to the incident wave: the cube has **scattered** some of the incident light, due to radiation by the induced dipole moment.

Scattering of electromagnetic plane-wave power by a dielectric sphere



Total scattering cross section of a dielectric sphere (continued)

It is customary to relate the total power in scattered light to the intensity of incident light by

$$P_{\text{scattered}} = \sigma_{sc} I_I \quad ,$$

where σ_{sc} is the **total scattering cross section** of our dielectric sphere.

- σ_{sc} has units of area; one can think of it as something like the area of the shadow cast by the dielectric sphere.
- Note, however, that in general this area can be very different from the *geometrical* cross-sectional area of the object that does the scattering.

Total scattering cross section of a dielectric sphere (continued)

In the present case of the dielectric sphere,

$$\langle P \rangle = \left(\frac{4\pi}{3}\right)^2 \frac{a^6 \chi_e^2 \omega^4}{3c^3} E_0^2 = \sigma_{sc} I_I = \sigma_{sc} \frac{c}{8\pi} E_0^2 \quad , \text{ or}$$

$$\sigma_{sc} = 2\left(\frac{4\pi}{3}\right)^3 \frac{a^6 \chi_e^2 \omega^4}{c^4} \quad .$$

- ❑ Note that it's not πa^2.
- ❑ Note especially the strong dependence on angular frequency. This will be the crucial point of our discussion next lecture.