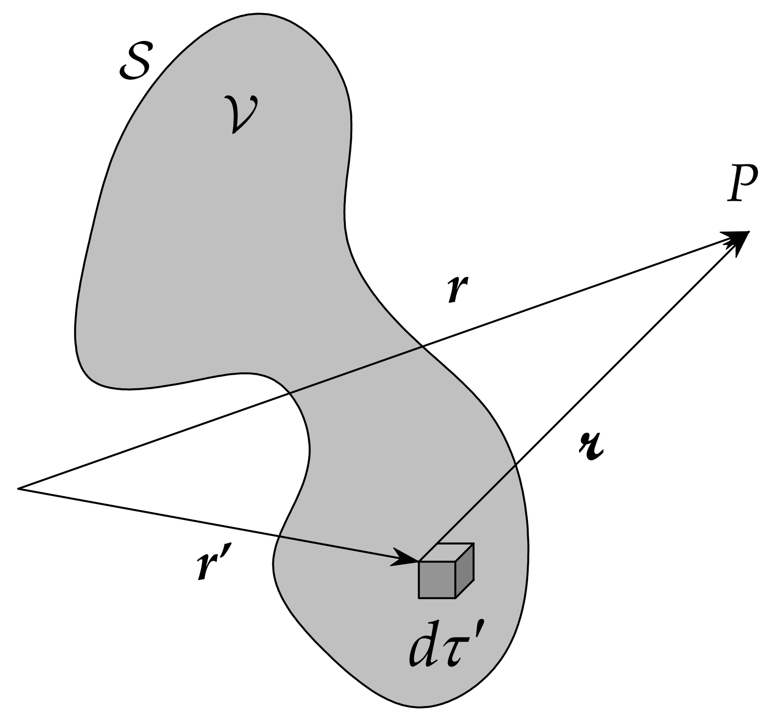
Today in Physics 218: radiating systems

- Multipole expansion for the potentials in radiating systems
- Radiation field in the dipole approximation
- Radiation by accelerating charges: the Larmor formula

The Crab Nebula, M1 – the remnant of the supernova of 1054 – emits much of its visible radiation by acceleration of electrons in magnetic fields. (Palomar Observatory photo.)

Multipole expansions for potentials in radiating systems

It's time to move on to more complicated radiating systems than dipoles, so let's go back to our volume full of charges and currents, and ask again what the potentials and fields are at some location $\boldsymbol{r}$ in the far field. Start with:



$$V(\boldsymbol{r},t)=\int_{\mathcal{V}}\frac{\rho\left(\boldsymbol{r}',t-\boldsymbol{\imath}/c\right)}{\boldsymbol{\imath}}d\tau'$$

where, as usual, $\boldsymbol{\imath}=\sqrt{r^2+r'^2-2\boldsymbol{r}\cdot\boldsymbol{r}'}=r\sqrt{1-\frac{2\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2}+\frac{r'^2}{r^2}}$.

Multipole expansions for potentials in radiating systems (continued)

If we're a long way from the charges and currents $(\mathbf{r}' \ll \mathbf{r})$, we can expand $\mathscr{r}$ in a series and make a first-order approximation:

$$\mathscr{r} = r\sqrt{1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}} \cong r\sqrt{1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2}}$$

Use $(1+x)^n$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$$

$$\cong r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \quad ;$$

to first order in $\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}$

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \quad .$$

also to first order in $\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}$

Thus
$$\rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right) \cong \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{r}\cdot\mathbf{r}'}{rc}\right) \quad .$$

Multipole expansions for potentials in radiating systems (continued)

Since the $\mathbf{r}\cdot\mathbf{r}'$ term is much smaller than the other two terms in the time argument, we can fruitfully expand in a Taylor series about $t_0 = t - r/c$ to make another first-order approximation:

$$\rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right) = \rho\left(\mathbf{r}', t_0\right) + \dot{\rho}\left(\mathbf{r}', t_0\right)\frac{\mathbf{r}\cdot\mathbf{r}'}{rc} + \frac{1}{2!}\dot{\rho}\left(\mathbf{r}', t_0\right)\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}\right)^2 + \ldots$$

where the dots indicate differentiation by time. Suppose that the system is oscillating, so that $\rho\left(\mathbf{r}', t\right) = \rho_0\left(\mathbf{r}'\right)\sin\omega t$:

$$\rho\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right) = \rho_0\left(\mathbf{r}'\right)\sin\omega t + \omega\rho_0\left(\mathbf{r}'\right)\cos\omega t\,\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}$$

$$- \frac{\omega^2}{2!}\rho_0\left(\mathbf{r}'\right)\sin\omega t\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}\right)^2 + \ldots$$

Multipole expansions for potentials in radiating systems (continued)

Thus the ratio of amplitudes of successive time derivatives is ω, and the ratio of successive terms in the series is $r'\omega/c$. In the far field, $r'\omega/c \ll 1$, so a first-order approximation for the charge density is appropriate:

$$\rho\left(\mathbf{r}', t - \frac{\boldsymbol{\mathscr{r}}}{c}\right) \cong \rho\left(\mathbf{r}', t_0\right) + \dot{\rho}\left(\mathbf{r}', t_0\right)\frac{\mathbf{r} \cdot \mathbf{r}'}{rc} \quad ,$$

and thus to first order in r'/r and $r'\omega/c$, the retarded scalar potential is

$$V(\mathbf{r}, t) = \int_{\mathcal{V}} \left(\rho + \dot{\rho}\frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right)\left(\frac{1}{r}\right)\left(1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right) d\tau'$$

Multiply it out:

Multipole expansions for potentials in radiating systems (continued)

$$V(\mathbf{r},t) \cong \frac{1}{r}\int_{\mathcal{V}} \rho(\mathbf{r}',t_0)\,d\tau' + \frac{\mathbf{r}}{r^3}\cdot\int_{\mathcal{V}} \mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau'$$

$$+\frac{\mathbf{r}}{r^2 c}\cdot\int_{\mathcal{V}} \mathbf{r}'\dot{\rho}(\mathbf{r}',t_0)\,d\tau' + \cancel{\frac{1}{c}\int_{\mathcal{V}} \dot{\rho}(\mathbf{r}',\mathbf{r}',t_0)\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^2 d\tau'} \quad .$$

Second order!

Now let $Q = \int \rho\, d\tau'$ and $p = \int \mathbf{r}'\rho\, \boldsymbol{d}\tau'$, and note that

$$\dot{\rho}(t_0) = \frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial t_0}\frac{\partial t_0}{\partial t} = \frac{\partial\rho}{\partial t_0} \quad :$$

$$\boxed{V(\mathbf{r},t) \cong \frac{Q}{r} + \hat{\mathbf{r}}\cdot\frac{\mathbf{p}(t-r/c)}{r^2} + \hat{\mathbf{r}}\cdot\frac{\dot{\mathbf{p}}(t-r/c)}{rc} \quad .}$$

Multipole expansions for potentials in radiating systems (continued)

These are the first terms in the multipole expansion of radiating systems – through the dipole terms. For quadrupole terms, keep through second order in r'/r and $r'\omega/c$.

Some features of this result:

- The charge term, Q/r, is constant (charge within $\mathcal{V}$ is constant!), so it doesn't lead to radiation.
- Both of the dipole terms do, though. In the far field, the second term is larger than the first:

$$\hat{\boldsymbol{r}} \cdot \frac{\boldsymbol{p}(t-r/c)}{r^2} \ll \hat{\boldsymbol{r}} \cdot \frac{\dot{\boldsymbol{p}}(t-r/c)}{rc} \quad \text{in the far field,}$$

  just as was true for the similar terms in the pure-dipole case (see lecture notes for 5 March 2004).

Multipole expansions for potentials in radiating systems (continued)

We can apply the same reasoning to the retarded vector potential $\boldsymbol{A}$:

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{1}{c}\int_{\mathcal{V}}\frac{\boldsymbol{J}(\boldsymbol{r}',t-\boldsymbol{\mathscr{r}}/c)}{\boldsymbol{\mathscr{r}}}d\tau'\cong\frac{1}{c}\int_{\mathcal{V}}\left(\boldsymbol{J}+\dot{\boldsymbol{J}}\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2}\right)\left(\frac{1}{r}\right)\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2}\right)d\tau' \quad .$$

The first term we get when we multiply this out is

$$\frac{1}{rc}\int_{\mathcal{V}}\boldsymbol{J}(\boldsymbol{r}',t-\boldsymbol{\mathscr{r}}/c)d\tau' \quad .$$

We showed, in PHY 217 (problem 5.7) that

$$\int_{\mathcal{V}}\boldsymbol{J}(\boldsymbol{r}',t)d\tau'=\frac{d\boldsymbol{p}}{dt} \quad ,$$

where $\boldsymbol{p}$ is the total dipole moment for everything in $\mathcal{V}$.

Flashback: solution to problem 5.7

We are given a hint to evaluate $\int_{\mathcal{V}} \nabla \cdot (x\boldsymbol{J})\, d\tau$:

$$\int_{\mathcal{V}} \nabla \cdot (x\boldsymbol{J})\, d\tau = \int_{\mathcal{V}} x\nabla \cdot \boldsymbol{J} d\tau + \int_{\mathcal{V}} \boldsymbol{J} \cdot \nabla x d\tau \quad \text{(Product Rule \#5)}$$

$$\oint_{\mathcal{S}} x\boldsymbol{J} \cdot d\boldsymbol{a} = -\int_{\mathcal{V}} x \frac{\partial \rho}{\partial t} d\tau + \int_{\mathcal{V}} J_x d\tau \qquad \text{(div. thm., continuity)}$$

$$0 = -\frac{d}{dt} \int_{\mathcal{V}} \rho x d\tau + \int_{\mathcal{V}} J_x d\tau \qquad (\boldsymbol{J} = 0 \text{ on } \mathcal{S})$$

$$\int_{\mathcal{V}} J_x d\tau = \frac{d}{dt} p_x \quad .$$

Flashback: solution to problem 5.7 (continued)

Similarly,

$$\int_{\mathcal{V}} J_y \, d\tau = \frac{d}{dt} p_y \quad , \quad \int_{\mathcal{V}} J_z \, d\tau = \frac{d}{dt} p_z \quad .$$

Multiply these three results by their corresponding unit vectors and add them up:

$$\int_{\mathcal{V}} \boldsymbol{J} \, d\tau = \frac{d}{dt} \boldsymbol{p} \quad \text{(Q.E.D.)}.$$

Multipole expansions for potentials in radiating systems (continued)

Thus the first term in the expansion for $\boldsymbol{A}$ is

$$\frac{1}{rc}\int_{\mathcal{V}} \boldsymbol{J}\left(\boldsymbol{r}', t - \boldsymbol{\imath}/c\right) d\tau' = \frac{1}{rc}\dot{\boldsymbol{p}} \quad .$$

All the other terms contain higher powers of r'/r than this term, and are therefore smaller. These other terms are similar in size to the terms we neglected in the expansion of the scalar potential. Thus, to the same order of approximation as the above expression for V,

$$\boxed{\boldsymbol{A}\left(\boldsymbol{r}, t\right) \cong \frac{1}{rc}\dot{\boldsymbol{p}} \quad .}$$

(Just as in magnetostatics, there's no monopole term, radiating or not.)

Radiation field in the dipole approximation

Now we can compute the fields at point $\boldsymbol{r}$:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\cong \cancel{\frac{Q}{r^2}} - \nabla\left(\cancel{\hat{\boldsymbol{r}}\cdot\frac{\boldsymbol{p}(t-r/c)}{r^2}}\right) - \nabla\left(\hat{\boldsymbol{r}}\cdot\frac{\dot{\boldsymbol{p}}(t-r/c)}{rc}\right) - \frac{\ddot{\boldsymbol{p}}(t-r/c)}{rc^2} \quad .$$

No radiation

Smaller than the next term, in the far field

$$\Rightarrow \quad \boldsymbol{E}_{\mathrm{rad}}(\boldsymbol{r},t) \cong -\nabla\left(\hat{\boldsymbol{r}}\cdot\frac{\dot{\boldsymbol{p}}(t-r/c)}{rc}\right) - \frac{\ddot{\boldsymbol{p}}(t-r/c)}{rc^2} \quad .$$

Use the chain rule:

$$\nabla = (\nabla t_0)\frac{\partial}{\partial t_0} = \left(\nabla\left[t-r/c\right]\right)\frac{\partial}{\partial t} = -\frac{\hat{\boldsymbol{r}}}{c}\frac{\partial}{\partial t} \quad ;$$

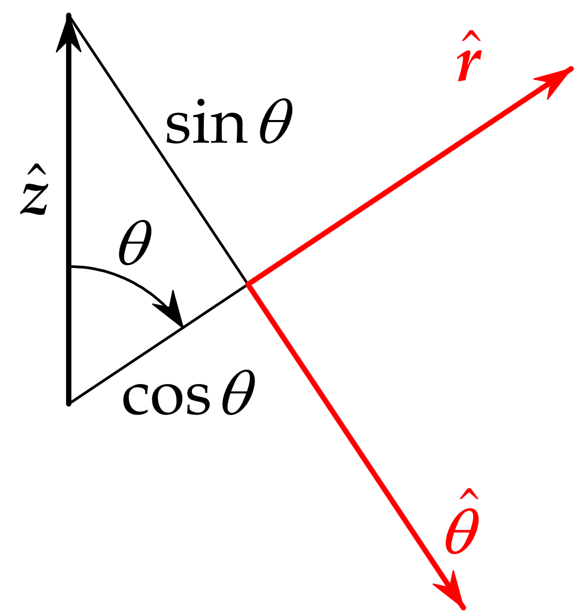
Radiation field in the dipole approximation (continued)

$$\Rightarrow \quad \mathbf{E}_{\text{rad}}(\mathbf{r},t) \cong \frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t-r/c)}{rc^2}\hat{\mathbf{r}} - \frac{\ddot{\mathbf{p}}(t-r/c)}{rc^2} \quad .$$

Now for $\mathbf{B}$. Without loss of generality we can line the z axis up with $\ddot{\mathbf{p}}(t-r/c)$:

$$\mathbf{B}_{\text{rad}} = \nabla\times\mathbf{A} = -\frac{1}{rc^2}\left(-\ddot{p}_\phi\hat{\boldsymbol{\theta}} + \ddot{p}_\theta\hat{\boldsymbol{\phi}}\right)$$

$$= -\frac{1}{rc^2}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t-r/c) \quad .$$



Now use $\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$:

$$\ddot{\mathbf{p}} = \ddot{p}\hat{\mathbf{z}} = \ddot{p}\hat{\mathbf{r}}\cos\theta - \ddot{p}\hat{\boldsymbol{\theta}}\sin\theta \quad .$$

Radiation field in the dipole approximation (continued)

so

$$\boldsymbol{E}_{\mathrm{rad}}(\boldsymbol{r},t)=\frac{\ddot{p}\cos\theta}{rc^2}\hat{\boldsymbol{r}}-\frac{\ddot{p}\hat{\boldsymbol{r}}\cos\theta-\ddot{p}\hat{\boldsymbol{\theta}}\sin\theta}{rc^2}=\boxed{\frac{\ddot{p}}{c^2}\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}}\ ,$$

and $$\boldsymbol{B}_{\mathrm{rad}}=-\frac{\ddot{p}}{rc^2}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right)=\boxed{\frac{\ddot{p}}{c^2}\frac{\sin\theta}{r}\hat{\boldsymbol{\phi}}}\ .$$

Compare these results to those obtained in lecture on 5 March 2004 to see that this reproduces our previous results, for the case of a "pure" dipole.

Radiation by accelerating charges: the Larmor formula

The Poynting vector corresponding to these fields is

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}} \quad ,$$

and the total power emitted by the charges and currents within $\mathcal{V}$ is

$$P = \int \boldsymbol{S}\cdot d\boldsymbol{a}$$

Integrate over any area in the far field that completely encloses $\mathcal{V}$

$$= \int \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}r^2\sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}d\phi = \frac{1}{4\pi}\frac{\ddot{p}^2}{c^3}\left(\frac{4}{3}\right)(2\pi) = \boxed{\frac{2}{3}\frac{\ddot{p}^2}{c^3}} \quad .$$

Radiation by accelerating charges: the Larmor formula (continued)

Consider, as an example of the application of this fairly general formula, the case of an electric dipole, but a different one than we treated before.

- Two opposite charges connected with a spring, so that the dipole-moment variation comes from the motion of the charges rather than changes in the magnitude of the charges:

$$\boldsymbol{p}(t) = qz(t)\hat{\boldsymbol{z}}$$

$$\ddot{p}(t) = q\frac{d^2z}{dt^2}\hat{\boldsymbol{z}} = qa\hat{\boldsymbol{z}} \quad \Rightarrow \quad \boxed{P = \frac{2}{3}\frac{q^2a^2}{c^3}} \; .$$

Larmor formula

Electromagnetic radiation is generated by *accelerating* charges.