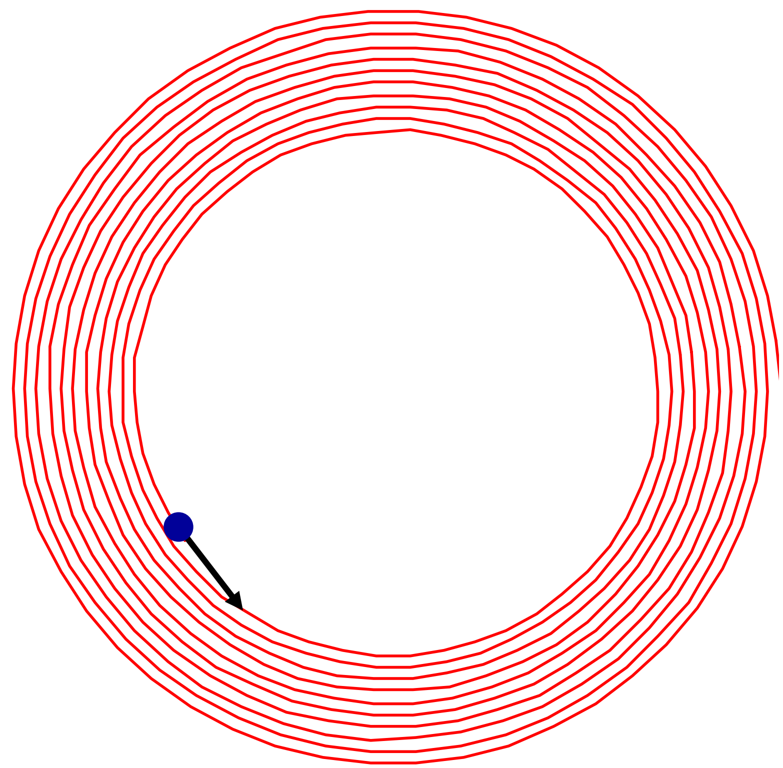
Today in Physics 218: radiation reaction

- ❑ Radiation reaction
- ❑ The Abraham-Lorentz formula; radiation reaction force



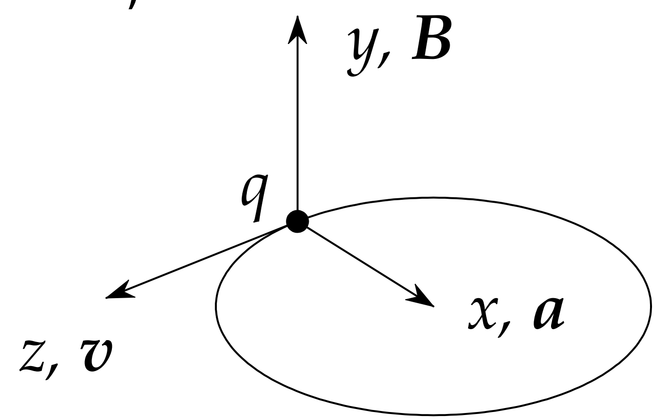
The path of the electron in today's first example (radial decay greatly exaggerated).

Synchrotron radiation

If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{a}$ (the other "simple" geometry), as in the case of uniform circular motion, P and $dP/d\Omega$ can be calculated with just a little more effort than the previous problem. (This in fact is problem !11.16 in the book, which will not be assigned.) The answers are

$$\left(\frac{dP}{d\Omega}\right)_{\text{emitted}} = \frac{q^2a^2}{4\pi c^3}\frac{(1-\beta\cos\theta)^2-\left(1-\beta^2\right)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5} \quad ,$$

$$P = \frac{2}{3}\frac{q^2a^2}{c^3}\gamma^4 \quad .$$



The coordinate system is described at right.

Synchrotron radiation (continued)

- The most common way to see charges in uniform circular motion in nature is of course to put some in motion in a uniform magnetic field.
- This result shows that the radiation still tends to be beamed in the forward direction.
- Charges used to be accelerated to high energies like this, in variable-$\boldsymbol{B}$ machines called **synchrotrons**, and the radiation resulting from the centripetal acceleration, for which the total power is given by the expression above, has been called synchrotron radiation ever since.
- Most of the radio radiation by normal galaxies is produced in this way, by electrons spiraling around in interstellar magnetic fields.

Radiation reaction

This is a very deep topic that, unlike everything else you've seen in electrodynamics, still contains many important unresolved problems, and many aspects that seem to indicate that electrodynamics might not be internally consistent. We will only touch on some of the important features.

Take a charge and a region of uniform magnetic field. Give the charge an impulse of kinetic energy, with its initial velocity lying in the plane perpendicular to $\boldsymbol{B}$. What happens?

- ❑ The charge moves in circles, at speed v and acceleration $a = v^2/r$.
- ❑ And it radiates, according to the formula just discussed.

Radiation reaction (continued)

- Radiation takes energy away from the charge. Where does this energy come from? Only one source exists: the charge's kinetic energy.
  - The uniform magnetic field is the only other energy density around, and magnetic fields can't do work on charges.
- So the radiation decreases the kinetic energy and speed of the particle, an effect known as **radiation reaction.**
- This decrease in kinetic energy can be characterized as work, and thus expressed in terms of a force that does the work.
  - And the force is what causes all the confusion…

Example: radiation reaction and cyclotron motion

Consider the charge we started with, given an impulse of kinetic energy and allowed to fly in "circles" in a uniform magnetic field. Describe the subsequent motion, and work out how long it takes the charge's speed to decay away.

Solution. Most radiation reaction problems are most easily and less confusingly solved with energy techniques. We'll assume the motion is instantaneously circular, and start with

$$P = \frac{2}{3}\frac{q^2}{c^3}a^2 = \frac{2}{3}\frac{q^2}{c^3}\left(\frac{qvB}{mc}\right)^2$$

$$= -\frac{dE}{dt} = -\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -mv\frac{dv}{dt} \quad .$$

This can be rearranged for integration:

Example: radiation reaction and cyclotron motion (continued)

$$\int_{v_0}^{v} \frac{dv'}{v'} = -\frac{2}{3}\frac{q^4 B^2}{m^3 c^5}\int_0^t dt'$$

$$\ln\left(\frac{v}{v_0}\right) = -\frac{2}{3}\frac{q^4 B^2}{m^3 c^5} t \quad .$$

$$v = v_0 \exp\left[-\frac{2}{3}\frac{q^4 B^2}{m^3 c^5} t\right] = v_0 e^{-t/\tau} \quad , \text{ where}$$

$$\tau = \frac{3m^3 c^5}{2q^4 B^2} \quad .$$

v_0

Example: radiation reaction and cyclotron motion (continued)

Some numbers may help: suppose that the magnetic field strength were $\boldsymbol{B}$ = 1 tesla = 10^4 gauss (a typical value in the laboratory) and the charge is an electron:

$$\tau = \frac{3m^3c^5}{2q^4B^2} = 5.16 \text{ sec} \quad .$$

Short, but the period of the electron's "orbit" in the magnetic field is much shorter:

$$F \cong \frac{qvB}{c} = ma \cong m\frac{v^2}{r}$$

cgs units, remember…

$$\Rightarrow \quad T = \frac{2\pi r}{v} = \frac{2\pi mc}{qB} = 3.57 \times 10^{-11} \text{ sec.}$$

Example: radiation reaction and cyclotron motion (continued)

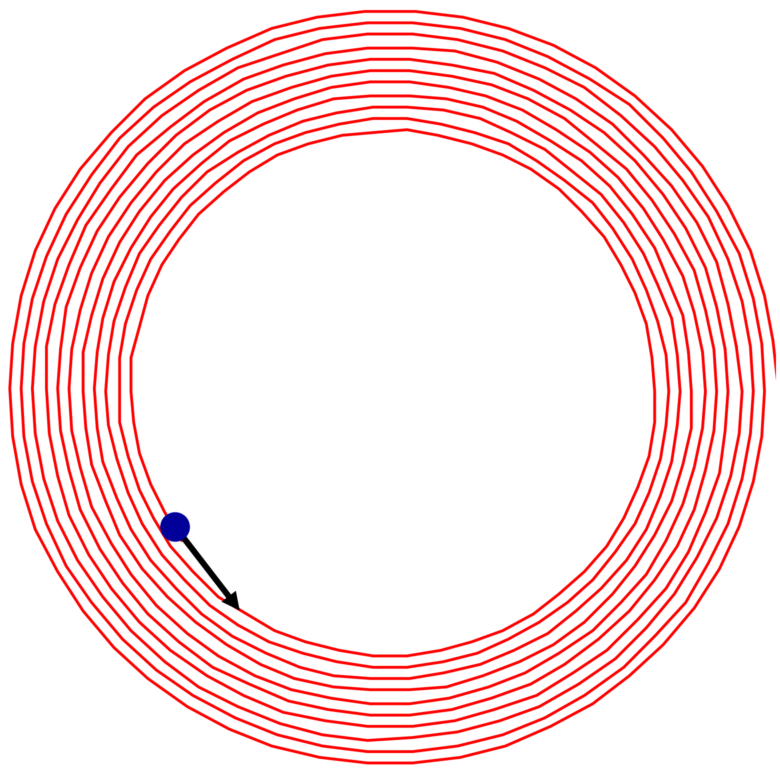
So the electron makes $T/\tau = 1.4\times 10^{11}$ orbits before the velocity has decayed to $1/e$ of its original value.

- This means the initial assumption of circular orbits is a very good approximation.
- The orbital radius, given as above by

$$r = \frac{mvc}{qB} \quad ,$$

decreases as the speed decreases: the electron spirals inward, as seen (with the radial decay exaggerated so as to render it visible) on the next page…

Example: radiation reaction and cyclotron motion (continued)



Example: radiation reaction and cyclotron motion (continued)

- A closely related problem is of great historical importance in physics.
  As you will show in Problem 11.14 on the next homework set, the electron in the Bohr model of the hydrogen atom, a point charge in orbit about a point nucleus, also exhibits this sort of radial decay. The time it takes the electron to spiral in to the nucleus, due to radiation losses in orbit, turns out to be very short. Atoms are thus unstable in the (otherwise quite successful) Bohr model. This led to the successful supposition of wave properties of the electron, which subsequently led to all of quantum mechanics as we know it.

Reaction force, and the Abraham-Lorentz formula

Here we will derive a formula for the force corresponding to radiation reaction in *non-relativistic* situations.

- Consider an electric charge q undergoing acceleration $a(t)$. At any instant it radiates power according to the Larmor formula:

$$P = \frac{2}{3}\frac{q^2 a^2(t)}{c^3} \quad .$$

- So if nothing else happens to the charge, it will not gain kinetic energy at the rate one might expect from the acceleration, on account of the flow of some of this energy into radiation. It's as if an extra force is at work:

Reaction force, and the Abraham-Lorentz formula (continued)

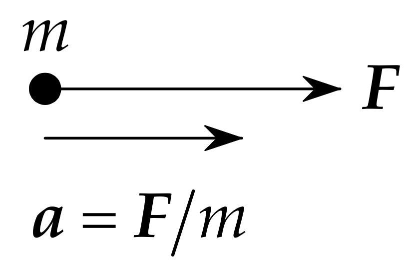
Supposing that the power in radiation is a result of work done against the extra force, we can find out how big it is:

Work done against $\boldsymbol{F}_{\text{rad}}$ =

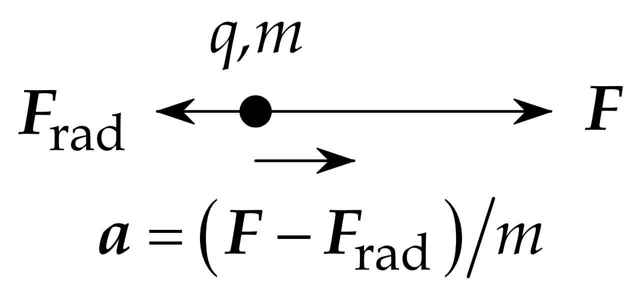
$$\int (\text{Power radiated})\, dt$$

$$-\int_{t_1}^{t_2} \boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v}\, dt = \frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2} a^2\, dt \quad .$$

Uncharged



Charged



Reaction force, and the Abraham-Lorentz formula (continued)

- Since $a^2 = \frac{d\boldsymbol{v}}{dt}\cdot\frac{d\boldsymbol{v}}{dt}$, the right-hand side of this expression can be integrated by parts:

$$-\int_{t_1}^{t_2} \boldsymbol{F}_{\text{rad}}\cdot\boldsymbol{v}\,dt = \frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2}\frac{d\boldsymbol{v}}{dt}\cdot\frac{d\boldsymbol{v}}{dt}\,dt$$

$$\boldsymbol{x} = d\boldsymbol{v}/dt\,,$$
$$d\boldsymbol{y} = \left(d\boldsymbol{v}/dt\right)dt;\ \boldsymbol{y} = \boldsymbol{v}$$

$$= \frac{2}{3}\frac{q^2}{c^3}\left[\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\boldsymbol{v}\cdot\frac{d^2\boldsymbol{v}}{dt^2}\,dt\right] \quad .$$

- Suppose that we are pushing the charge in such a way that it’s in the same state – $\boldsymbol{v}$ and $\boldsymbol{a}$ – at the initial and final times, but allow the state to change in between.

Reaction force, and the Abraham-Lorentz formula (continued)

- Then the integral on the right-hand side just represents the time average of a^2 between those times.
- It also means that

$$\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\bigg|_{t_1}^{t_2}=0 \quad ,$$

because $\boldsymbol{v}$ and $\boldsymbol{a}$ are the same at the two times. Thus

$$-\int_{t_1}^{t_2}\boldsymbol{F}_{\text{rad}}\cdot\boldsymbol{v}\,dt=-\frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2}\boldsymbol{v}\cdot\dot{\boldsymbol{a}}\,dt \quad . \qquad \dot{\boldsymbol{a}}=\frac{d^2\boldsymbol{v}}{dt^2}$$

- Equate the integrands:

$$\boxed{\boldsymbol{F}_{\text{rad}}=-\frac{2}{3}\frac{q^2}{c^3}\dot{\boldsymbol{a}} \quad .}$$

Abraham-Lorentz formula

How to use the radiation reaction force (continued)

You will get practice in the use of the radiation reaction force in Problem 11.17 on this week's homework. To help you get started…

Example (11.4 in the book): Calculate the radiation damping of a charged particle attached to a spring with natural frequency ω_0, driven at frequency ω.

Solution:

The radiation reaction force involves one more time derivative than is usual in kinematic problems, which often complicates otherwise simple situations. In this case periodic motion keeps this one from getting out of hand.

$$F = ma \quad \Rightarrow \quad F_{\text{spring}} + F_{\text{rad}} + F_{\text{driving}} = m\ddot{x}$$

How to use the radiation reaction force (continued)

$$-m\omega_0^2 x + \frac{2}{3}\frac{q^2}{c^3}\dddot{x} + F_0 \cos \omega t = m\ddot{x} \quad .$$

Have no fear. If the system is driven, it will oscillate at frequency ω. The worst the extra term can do is produce a (potentially complicated) phase delay. So

$$x(t) = x_0 \cos(\omega t + \delta)$$

$$\dot{x} = -\omega x_0 \sin(\omega t + \delta)$$

$$\ddot{x} = -\omega^2 x_0 \cos(\omega t + \delta)$$

$$\dddot{x} = \omega^3 \sin(\omega t + \delta) = -\omega^2 \dot{x} \quad (!),$$

and we're left with $-m\omega_0^2 x - \frac{2}{3}\frac{q^2\omega^2}{c^3}\dot{x} + F_0 \cos \omega t = m\ddot{x} \quad .$

How to use the radiation reaction force (continued)

Thus

$$\ddot{x} + \frac{2}{3}\frac{q^2\omega^2}{mc^3}\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \quad ;$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \quad .$$

We have solved this damped harmonic oscillator problem many times (for instance, in lecture on 18 February 2004):

$$x = x_0 e^{-i\omega t} \quad , \quad x_0 = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right) - i\omega\gamma} \quad .$$