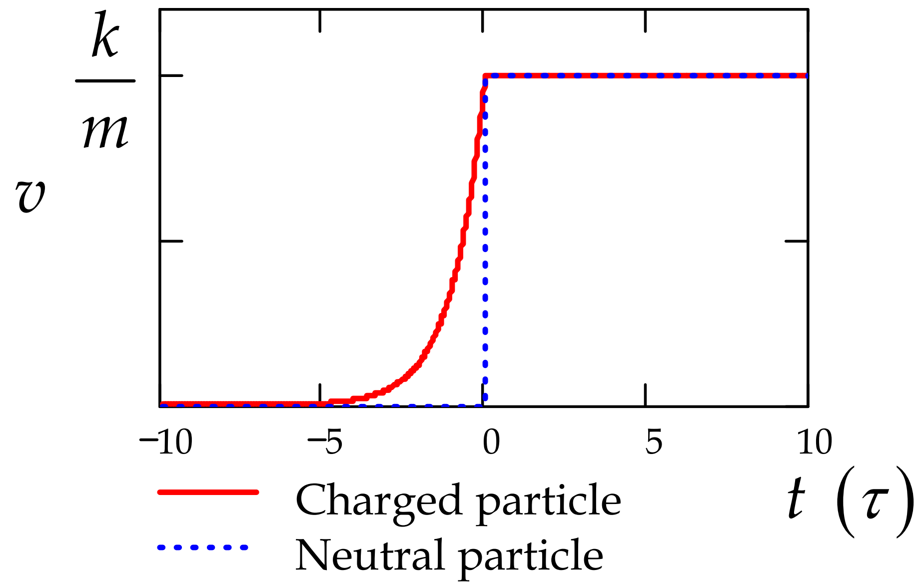
Today in Physics 218: radiation reaction II

- The nature of the radiation-reaction force; a fundamental inconsistency of electrodynamics.
- Other problems with the Abraham-Lorentz formula: runaway solutions and acausal "preaccelerations."



Preacceleration. Almost as bad as a runaway.

Where does the radiation-reaction force come from?

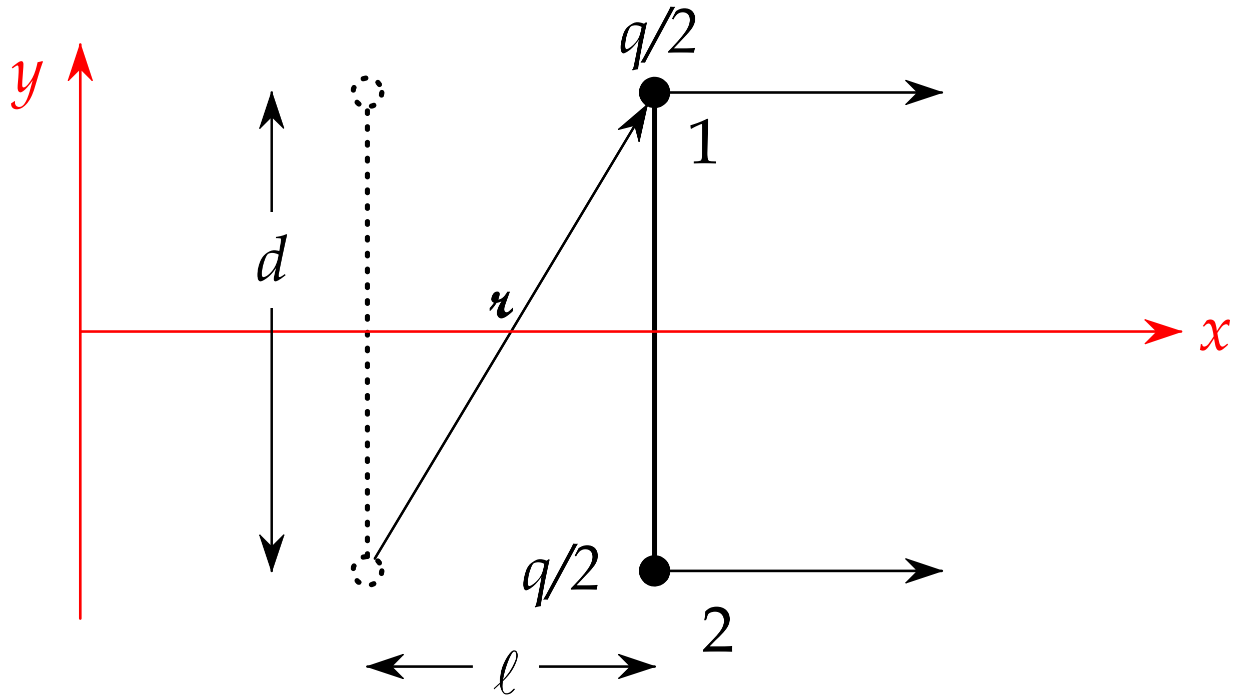
The short answer: it's the force on a charge from the fields of which it is the source.

- The problem: this is supposed to be zero, if Newton's third law is valid!

The simplest way to see this non-cancellation of internal and self-forces is to consider (as Griffiths does) the transverse-moving dumbbell. In this example the essential physics appears correctly, and the self-force obtained replicates the previous answer. We will sketch this derivation in the following.

- The appearance of the previous, plausible answer is accidental, as Griffiths mentions.

Where does the radiation-reaction force come from? (continued)



Consider two charges $q/2$, separated by distance d and moving as shown, but instantaneously at rest $(\boldsymbol{v} = 0, \boldsymbol{a} \neq 0)$ at retarded time t_r. Find the total force: the sum of the forces of each on the other, in the limit $d \to 0$.

Where does the radiation-reaction force come from? (continued)

For this we use the field we obtained from the Liénard-Wiechert potentials:

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{GC}} + \boldsymbol{E}_{\mathrm{rad}} = \frac{q\mathscr{r}}{2\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\left(c^2 - v^2\right)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times\left(\boldsymbol{u}\times\boldsymbol{a}\right)\right]$$

$$= \frac{q\mathscr{r}}{2\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)^3}\left[\boldsymbol{u}\left(c^2 + \boldsymbol{\mathscr{r}}\cdot\boldsymbol{a}\right) - \boldsymbol{a}\left(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u}\right)\right] \quad ,$$

where $\quad \boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v} = c\hat{\boldsymbol{\mathscr{r}}} \quad , \quad \boldsymbol{\mathscr{r}} = \ell\hat{\boldsymbol{x}} + d\hat{\boldsymbol{y}} \quad ;$ so

$\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u} = c\mathscr{r}$, $\boldsymbol{\mathscr{r}}\cdot\boldsymbol{a} = \ell a$, and $\mathscr{r} = \sqrt{\ell^2 + d^2}$.

Where does the radiation-reaction force come from? (continued)

Thus,

$$\boldsymbol{E}(\text{at 1 from 2}) = \frac{q\boldsymbol{\mathscr{r}}}{2(c\mathscr{r})^3}\left[c\hat{\boldsymbol{\mathscr{r}}}\left(c^2+\ell a\right)-\boldsymbol{a}(c\mathscr{r})\right]$$

$$= \frac{q}{2c^2\left(\ell^2+d^2\right)^{3/2}}\left[\boldsymbol{\mathscr{r}}\left(c^2+\ell a\right)-\mathscr{r}^2\boldsymbol{a}\right] \quad .$$

$$E_x(\text{at 1 from 2}) = \frac{q}{2c^2\left(\ell^2+d^2\right)^{3/2}}\left[\ell\left(c^2+\ell a\right)-\left(\ell^2+d^2\right)a\right]$$

$$= \frac{q\left(\ell c^2 - a d^2\right)}{2c^2\left(\ell^2+d^2\right)^{3/2}} = E_x(\text{at 2 from 1}) \quad ,$$

Where does the radiation-reaction force come from? (continued)

Furthermore,

$$E_y\left(\text{at 1 from 2}\right) = \frac{qd\left(c^2 + \ell a\right)}{2c^2\left(\ell^2 + d^2\right)^{3/2}} = -E_y\left(\text{at 2 from 1}\right) \quad .$$

So the y components of the forces between 1 and 2 cancel, but the x components add.

We also have still to include the forces of the two charges on themselves. In this case

$$\boldsymbol{u} = c\hat{\boldsymbol{\imath}} - \boldsymbol{v} = c\hat{\boldsymbol{x}} \quad , \quad \boldsymbol{\imath} = \ell\hat{\boldsymbol{x}} \quad ; \text{ so}$$

$$\boldsymbol{\imath}\cdot\boldsymbol{u} = \ell c, \; \boldsymbol{\imath}\cdot\boldsymbol{a} = \ell a, \text{ and } \imath = \ell \; ,$$

Where does the radiation-reaction force come from? (continued)

so
$$\boldsymbol{E}(\text{on 1 from 1}) = \frac{q}{2(\ell c)^3}\left[c\ell\left(c^2+\ell a\right) - \ell a(\ell c)\right]\hat{\boldsymbol{x}} = \frac{q}{2\ell^2}\hat{\boldsymbol{x}}$$
$$= \boldsymbol{E}(\text{on 2 from 2}) \quad ,$$

and

$$\boldsymbol{F}_{\text{self}} = \frac{q}{2}E_x(\text{at 1 from 2})\hat{\boldsymbol{x}} + \frac{q}{2}E_x(\text{at 2 from 1})\hat{\boldsymbol{x}}$$
$$+\frac{q}{2}E_x(\text{at 1 from 1})\hat{\boldsymbol{x}} + \frac{q}{2}E_x(\text{at 2 from 2})\hat{\boldsymbol{x}}$$
$$= \frac{q^2\left(\ell c^2 - ad^2\right)}{2c^2\left(\ell^2+d^2\right)^{3/2}}\hat{\boldsymbol{x}} + \frac{q^2}{2\ell^2}\hat{\boldsymbol{x}} \quad .$$

Where does the radiation-reaction force come from? (continued)

Now we must expand this result in powers of d, so that we can let $d \to 0$, and be left with just the zeroth order term and lower, and a point charge q. This is not easy, and involves a series reversion; it's also not very illuminating, so we'll skip:

$$\begin{aligned}\boldsymbol{F}_{\text{self}} &= q^2\left[-\frac{a(t)}{4c^2 d} + \frac{2\dot{a}(t)}{3c^3} + O(d) + \ldots\right]\hat{\boldsymbol{x}} \\ &= \left[-\Delta m\boldsymbol{a}(t) + \boldsymbol{F}_{\text{rad}} + \hat{\boldsymbol{x}}O(d) + \ldots\right] \quad .\end{aligned}$$

- The first term can just be moved to the other side of the equation: it is the extra inertia from the potential energy of the two charges. It's as if the total mass were

$$m = 2m_0 + \Delta m = 2m_0 + \frac{\left(q^2/4d\right)}{c^2} \quad .$$

Where does the radiation-reaction force come from? (continued)

- This mass correction seems to make sense, but that's a lucky consequence of the geometry:
  - For a dumbbell oriented along x, it arrives with an extra factor of 1/2; for a sphere, the term has an extra factor of 4/3.
  - It turns out that if you express everything carefully in a Lorentz-transformation-invariant fashion, and take the nonrelativistic limit, all the factors turn back to 1 (see Jackson, second edition, section 17.5).
- The second term is the same as the Abraham-Lorentz radiation-reaction force, here identified as an imbalance of internal forces in a charge distribution.

Where does the radiation-reaction force come from? (continued)

- Taking the limit $d \to 0$ kills off all the terms of order d and higher, and turns the dumbbell into a point charge. But it also makes the Δm term blow up!
  - This problem remains even in quantum electrodynamics. In that field it is rendered harmless by the ruse of **mass renormalization**.
  - In classical electrodynamics, the way out is to suppose that the mass is all electrostatic potential energy, and to impose a small but finite size, $r_0 = q^2 / mc^2$, called the charge's **classical radius**. For instance, the classical radius of the electron is $r_0 = e^2 / mc^2 = 2.82 \times 10^{-13}$ cm. The $O(d)$ terms are still neglible if d is reduced to the limit of the classical charge size.

Runaways and preacceleration

The Abraham-Lorentz formula leads directly to several puzzling results, some of which don't get resolved if (for example) one re-derives a fully relativistic or quantum-mechanical equivalent. The worst is your choice of two related problems: exponential "runaway" solutions, or violation of cause and effect.

- Consider

$$\boldsymbol{F}_{\text{external}} + \boldsymbol{F}_{\text{rad}} = \boldsymbol{F}_{\text{external}} + \frac{2}{3}\frac{q^2}{c^3}\dot{\boldsymbol{a}} = m\boldsymbol{a}$$

With no external force, the solution is

$$\int_{a_0}^{a} \frac{da'}{a'} = \frac{3c^3}{2q^2}\int_0^t dt' \quad \Rightarrow \quad a(t) = a_0 \exp\left(\frac{3c^3 t}{2q^2}\right) \quad .$$

Runaways and preacceleration (continued)

- So either $a_0 = 0$, or the acceleration increases exponentially with time. This is called a **runaway solution.**
- Runaways can easily be avoided, but this causes other problems. Consider, for example, the situation in problem 11.28 in the book:
  A charged particle is subjected to an impulse:
  $$F_{\text{ext}} = \frac{k}{m}\delta(t).$$
  Solve the equation of motion and show that you can eliminate the runaway solution, but only at the expense of having an acausal solution.

Runaways and preacceleration (continued)

- **Solution**: first, integrate the equation of motion over a small region around the origin:

$$a = \tau \dot{a} + \frac{k}{m}\delta(t) \qquad\qquad \tau \equiv \frac{2q^2}{3c^3}$$

$$\int_{-\varepsilon}^{\varepsilon} a(t)\,dt = v(\varepsilon) - v(-\varepsilon) = \tau \int_{-\varepsilon}^{\varepsilon} \frac{da}{dt}\,dt + \frac{k}{m}\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt$$

$$= \tau\left[a(\varepsilon) - a(-\varepsilon)\right] + \frac{k}{m} \quad .$$

If the velocity is continuous (as it must be), then

$$\tau\left[a(\varepsilon) - a(-\varepsilon)\right] + \frac{k}{m} = 0 \quad .$$

Runaways and preacceleration (continued)

When $t < 0$, $\quad a = \tau\dot{a} \quad \Rightarrow \quad a(t) = a_0 e^{t/\tau}$.
When $t > 0$, $\quad a = \tau\dot{a} \quad \Rightarrow \quad a(t) = b_0 e^{t/\tau}$.
But, as we just showed above,

$$a(\varepsilon) - a(-\varepsilon) = -\frac{k}{\tau m}$$

$$b_0 - a_0 = \qquad\qquad ,$$

so the general solution is

$$a(t) = \begin{cases} a_0 e^{t/\tau} & (t < 0), \\ \left(a_0 - \dfrac{k}{\tau m}\right) e^{t/\tau} & (t > 0). \end{cases}$$

To eliminate the runaway we need $a_0 = k/\tau m$.

Runaways and preacceleration (continued)

But then,

$$a(t) = \begin{cases} a_0 e^{t/\tau} & (t < 0), \\ 0 & (t > 0). \end{cases}$$

The velocity in this case is

$$v(t < 0) = \int_{-\infty}^{t} a(t)\,dt = \frac{k}{\tau m} \int_{-\infty}^{t} e^{t/\tau}\,dt = \frac{k}{m} e^{t/\tau} \quad ;$$

$$v(t > 0) = v(0) + \int_{-\infty}^{t} a(t)\,dt = v(0) = \frac{k}{m} \quad .$$

Thus the name **preacceleration**; the velocity starts increasing before the force is applied.

Runaways and preacceleration (continued)

- Compare this to an uncharged particle, for which there's no radiation reaction force in the equations of motion:

$$a(t) = \frac{k}{m}\delta(t)$$

$$v(t) = \begin{cases} 0 & (t < 0), \\ \dfrac{k}{m} & (t > 0). \end{cases}$$