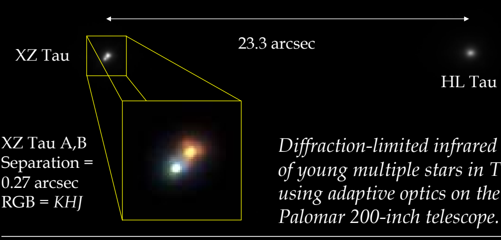
Today in Physics 218: diffraction

- Fields as sources of radiation: Huygens's principle.
- The Kirchhoff integral: "the far field is the Fourier transform of the near field."



Diffraction-limited infrared images of young multiple stars in Taurus, using adaptive optics on the Palomar 200-inch telescope.

Fields as sources: Huygens's principle

Sometimes one has a distribution of fields, instead of charges and currents, that can be considered the source of electromagnetic radiation propagating through space. In this case propagation delays need carefully to be accounted, as before, but instead of using retarded time and retarded length we will need just retarded length.

- And one building block: a spherical solution to the homogeneous wave equations for the fields.
- Consider one polarization component of an electric field $\boldsymbol{E}$. Treat this component as a scalar. It obeys

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E$$

Huygens's principle (continued)

- It turns out that the function

$$E(r,t) = \frac{\eta_0}{r}e^{i(kr-\omega t)} \quad \text{, where}$$

$$\boldsymbol{k} = k\hat{\boldsymbol{r}} \quad ,$$

  is an outward-propagating, spherical solution to the wave equation. The constant η_0, called the source strength, is related to the electric field at the location we consider to be the "source" fields. (See also problem 9.33 in Griffiths.)
- This is easily demonstrated: on the right-hand side of the wave equation we have

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{\eta_0 e^{ikr}}{c^2 r}\frac{d^2}{dr^2}e^{-i\omega t} = \frac{-\omega^2\eta_0 e^{ikr-i\omega t}}{c^2 r} = -k^2E(r,t) \quad ,$$

Huygens's principle (continued)

and on the left,

$$\nabla^2 E = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} E(r,t) = \frac{\eta_0 e^{-i\omega t}}{r^2}\frac{d}{dr} r^2 \left(-\frac{e^{ikr}}{r^2} + \frac{ike^{ikr}}{r} \right)$$

$$= \frac{\eta_0 e^{-i\omega t}}{r^2}\left(-ike^{ikr} + ike^{ikr} - k^2 re^{ikr} \right) = -k^2 E(r,t) \quad \text{, q.e.d.}$$

- We will treat the propagation of radiation from the "source" fields by considering each element of the field distribution to be a source of spherical waves, then adding up all the waves. This procedure is known as **Huygens's principle**, and the result will be a scalar (actually, single-polarization) account of **diffraction**.

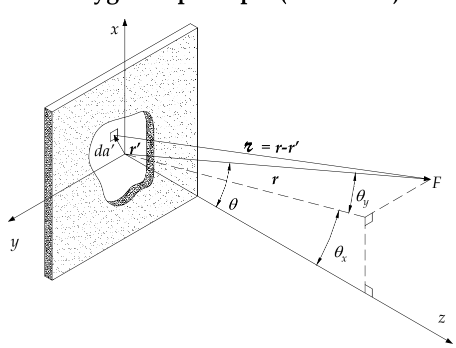
Huygens's principle (continued)

Consider a plane electromagnetic wave incident normally on a hole in an otherwise opaque screen. Some of the wave will pass through the hole and continue to propagate. What will be the electric field due to the light that got through, at some point *F* a great distance away from the hole?

- The normal approach to this problem is to note that the hole can considered to have a constant electric field at some instant in time, and for each infinitesimal element of it to be an independent source of spherical waves of the form we just described:

$$dE_F = \frac{\mathcal{E}_A(x', y')da'}{\mathscr{r}} e^{i(k\mathscr{r} - \omega t)}$$

Huygens's principle (continued)



Huygens's principle (continued)

- Since we have only one polarization, we can integrate this expression over the aperture straightforwardly to obtain E_F. Each element of the aperture lies a different distance from point F, and the phase differences between the spherical waves arising from these pathlength differences will provide constructive or destructive interference.
- From the diagram,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + z^2} = r\sqrt{1 - 2\frac{xx' + yy'}{r^2} + \frac{x'^2 + y'^2}{r^2}} \quad .$$

Huygens's principle (continued)

- Let's suppose point F lies in the far field $(r \gg \text{ any } r')$; then we can use a first-order approximation for $\mathscr{r}$:

$$\mathscr{r} \cong r\sqrt{1 - 2\frac{xx' + yy'}{r^2}} \cong r\left(1 - \frac{xx' + yy'}{r^2}\right) \quad ,$$

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r\left(1 - \frac{xx' + yy'}{r^2}\right)} \cong \frac{1}{r}\left(1 + \frac{xx' + yy'}{r^2}\right) \quad .$$

Thus $$\frac{e^{ik\mathscr{r}}}{\mathscr{r}} \cong e^{ikr} e^{-ik(xx' + yy')/r} \frac{1}{r}\left(1 + \frac{xx' + yy'}{r^2}\right)$$

$$\cong \frac{e^{ikr} e^{-ik(xx' + yy')/r}}{r} \quad .$$

to first order in r'/r.

Huygens's principle (continued)

- This leaves us with

$$E_F = \frac{e^{i(kr - \omega t)}}{r} \int_{\text{aperture}} \mathcal{E}_A(x', y') e^{-ik(xx' + yy')/r} da' \quad ,$$

or, with $k_x = k\theta_x \cong kx/r \quad , \quad k_y = k\theta_y \cong ky/r \quad ,$

$$E_F\left(k_x, k_y\right) = \frac{e^{i(kr - \omega t)}}{r} \iint_{\text{aperture}} \mathcal{E}_A(x', y') e^{-i\left(k_x x' + k_y y'\right)} dx'dy'$$

- We can define $\mathcal{E}_A(x', y')$ such that the aperture is built in:

$$\mathcal{E}'_A(x', y') = \mathcal{E}_A(x', y') \qquad \text{inside, and}$$
$$= 0 \qquad \text{outside the aperture,}$$

Huygens's principle (continued)

- so that

$$E_F\left(k_x,k_y\right)=\frac{e^{i(kr-\omega t)}}{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{E}'_A(x',y')e^{-i\left(k_x x'+k_y y'\right)}dx'dy' \quad .$$

Apart from the leading factor, this looks like the two-dimensional Fourier transform of the source strength per unit area, $\mathcal{E}'_A(x',y')$.

Reminder: Fourier transforms

One dimensional Fourier transform and inverse transform:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-ixs}ds \quad , \quad F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{ixs}dx \quad .$$

For two dimensions:

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(s,t)e^{-i(xs+yt)}dsdt \quad ,$$

$$F(s,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i(xs+yt)}dxdy \quad .$$

Reminder: Fourier transforms (continued)

With these definitions, it can be shown (and has been, in MTH 281) that *f* and *F* are related by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(s,t)\right|^2dsdt \quad ,$$

Rayleigh's theorem

a fact important in electrodynamics, as there are many ways to express energy conservation by its use.

- In the present case, the application of Rayleigh's theorem and energy conservation will enable us to identify the hitherto mysterious source strength $\mathcal{E}'_A(x',y')$ with more familiar physical quantities.

The Kirchhoff integral

We can group the factors in our last result to make it look exactly like a Fourier transform:

$$E_F\left(k_x,k_y\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2\pi e^{i(kr-\omega t)}\mathcal{E}'_A(x',y')}{r}e^{-i\left(k_x x'+k_y y'\right)}dx'dy',$$

and then apply Rayleigh's theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E_F\left(k_x,k_y\right)\right|^2 dk_x dk_y=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{2\pi e^{i(kr-\omega t)}\mathcal{E}'_A(x',y')}{r}\right|^2 dx'dy'$$

$$=\frac{4\pi^2}{r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{E}'_A(x',y')\right|^2 dx'dy' \quad .$$

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The Kirchhoff integral (continued)

We can also change integration variables (back!) on the left side:

$$dk_x dk_y=\frac{kdx}{r}\frac{kdy}{r}=\frac{4\pi^2}{\lambda^2 r^2}dxdy \quad ,$$

to obtain

$$\frac{4\pi^2}{\lambda^2 r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E_F\right|^2 dxdy=\frac{4\pi^2}{r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{E}'_A(x',y')\right|^2 dx'dy' \quad \text{, or}$$

$$\frac{c}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E_F\right|^2 dxdy=\frac{c}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\lambda\mathcal{E}'_A(x',y')\right|^2 dx'dy' \quad .$$

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The Kirchhoff integral (continued)

- The left-hand side is clearly the *power* in electromagnetic radiation passing through an *x,y* planar surface that runs through point *F*.
- By conservation of energy, the right-hand side must therefore be the power passing through the aperture. So we identify the source strength per unit area with the electric field in the plane of the aperture:

$$E_N\left(x',y'\right)=\lambda\mathcal{E}'_A(x',y')e^{-i\omega t} \quad ,$$

so that

$$\frac{c}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E_F\right|^2 dxdy=\frac{c}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|E_N\right|^2 dx'dy'$$

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The Kirchhoff integral (continued)

Thus, returning to our original result from Huygens's principle, we can write

$$\boxed{E_F\left(k_x,k_y,t\right)=\frac{e^{ikr}}{\lambda r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_N(x',y',t)e^{-i\left(k_x x'+k_y y'\right)}dx'dy'} \quad .$$

Kirchhoff integral

This is one of the fundamental relations of physical optics. Its meaning: the far field, E_F, and the "near field" E_N, distributed over the aperture, are essentially Fourier transforms of one another.