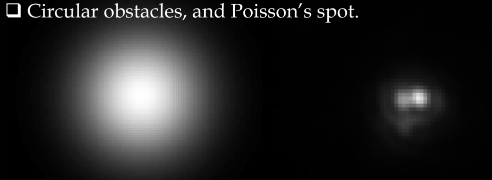
Today in Physics 218: diffraction by a circular aperture or obstacle

- ❑ Circular-aperture diffraction and the Airy pattern
- ❑ Circular obstacles, and Poisson's spot.



V773 Tau: AO off (and brightness turned way up)

V773 Tau: AO on

Neptune orbit diameter, seen from same distance

The circular aperture

Most experimental situations in optics (e.g. telescopes) have circular apertures, so the application of the Kirchhoff integral to diffraction from such apertures is of particular interest.

We start with a plane wave incident normally on a circular hole with radius a in an otherwise opaque screen, and ask: what is the distribution of the intensity of light on a screen a distance $R >> a$ away? The field in the aperture is constant, spatially:

$$E_N(x', y', t) = E_{N0}e^{-i\omega t} \quad ,$$

and the geometry is as follows:

The circular aperture (continued)

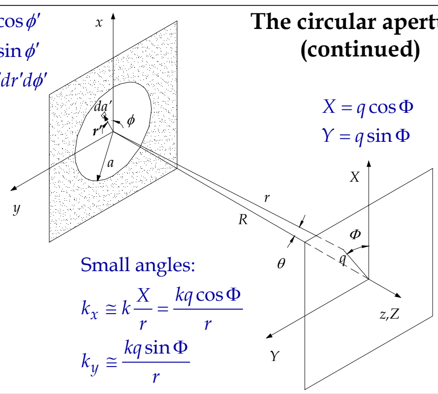
$x' = r'\cos\phi'$

$y' = r'\sin\phi'$

$da' = r'dr'd\phi'$

$X = q\cos\Phi$

$Y = q\sin\Phi$



Small angles:

$$k_x \cong k\frac{X}{r} = \frac{kq\cos\Phi}{r}$$

$$k_y \cong \frac{kq\sin\Phi}{r}$$

The circular aperture (continued)

Thus,

$$E_F\left(k_x,k_y,t\right)=\frac{e^{ikr}}{\lambda r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_N(x',y',t)e^{-i\left(k_x x'+k_y y'\right)}dx'dy'$$

$$=\frac{e^{ikr}}{\lambda r}\int_0^a dr'r'$$

$$\times\int_0^{2\pi}d\phi' E_{N0}e^{-i\omega t}\exp\left(-\frac{ikr'q}{r}\left(\cos\phi'\cos\Phi+\sin\phi'\sin\Phi\right)\right)$$

$$=\frac{E_{N0}e^{i(kr-\omega t)}}{\lambda r}\int_0^a dr'r'\int_0^{2\pi}d\phi'\exp\left(-\frac{ikr'q}{r}\cos\left(\phi'-\Phi\right)\right)\quad ,$$

The circular aperture (continued)

The aperture is symmetrical about the z axis, so we expect that the answer will be independent of the "screen" azimuthal coordinate Φ; without loss of generality, then, we can take $\Phi = 0$. The integral over ϕ' becomes

$$\mathcal{I}=\int_0^{2\pi}d\phi'\exp\left(-\frac{ikr'q}{r}\cos\phi'\right)\quad .$$

Don't try to integrate that directly; it's a Bessel function of the first kind, order zero:

$$J_0(-u)=J_0(u)=\frac{1}{2\pi}\int_0^{2\pi}e^{iu\cos v}dv\quad\Rightarrow\quad\mathcal{I}=2\pi J_0\left(\frac{kr'q}{r}\right)\quad .$$

Flashback: Bessel functions

The Bessel function of the first kind, of order m, can be represented by the integral

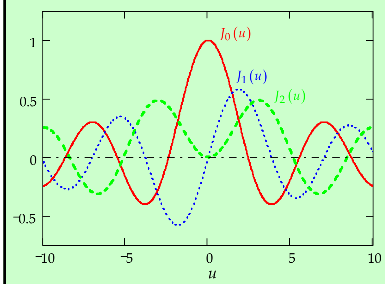
$$J_m(u)=\frac{i^{-m}}{2\pi}\int_0^{2\pi}e^{i(mv+u\cos v)}dv\quad .$$

Bessel functions of different order are related by the recurrence relation

$$\frac{d}{du}\left[u^m J_m(u)\right]=u^m J_{m-1}(u)\quad\Leftrightarrow\quad u^m J_m(u)=\int_0^u v^m J_{m-1}(v)\,dv\quad .$$

Recurrence relations of special functions are very useful when one has to integrate those special functions, as you're about to see.

Flashback: Bessel functions (continued)



Zeroes of the Bessel functions

J_0	J_1	J_2
2.405	0	0
5.520	**3.832**	5.136
8.654	7.016	8.417
11.792	10.174	11.620
14.931	13.324	14.796

The circular aperture (continued)

So the far field is

$$E_F(q,t) = \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \int_0^a dr' r' J_0\left(\frac{kqr'}{r}\right)$$

$$= \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \left(\frac{r}{kq}\right)^2 \int_0^{kaq/r} v J_0(v)\, dv \quad .$$

Now use the recurrence relation, with $m = 1$:

$$u J_1(u) = \int_0^u v J_0(v)\, dv \quad ;$$

$$E_F(q,t) = \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \left(\frac{r}{kq}\right)^2 \left(\frac{kaq}{r}\right) J_1\left(\frac{kaq}{r}\right) \quad .$$

The circular aperture (continued)

Rearrange the field in a somewhat more convenient form:

$$E_F(q,t) = \frac{\pi a^2 E_{N0} e^{i(kr-\omega t)}}{\lambda r} \frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right) \quad ,$$

or $$E_F(\theta,t) = \frac{E_{N0} A e^{i(kr-\omega t)}}{\lambda r} \frac{2J_1(ka\theta)}{ka\theta} \quad ,$$

whence $$I_F(ka\theta) = \frac{c}{8\pi} E_F(\theta,t) E_F^*(\theta,t) = \frac{c E_{N0}^2 A^2}{8\pi \lambda^2 r^2} \left[\frac{2J_1(ka\theta)}{ka\theta}\right]^2 \quad .$$

This leaves a minor problem: the expression is indeterminate at $ka\theta = 0$. But the recurrence relation can help us again:

The circular aperture (continued)

Take the recurrence relation at $m = 1$ and use the chain rule:

$$uJ_0(u) = \frac{d}{du}\left[uJ_1(u)\right] = J_1(u) + u\frac{dJ_1}{du}(u)$$

$$J_0(u) = \frac{dJ_1}{du}(u) + \frac{J_1(u)}{u} \quad .$$

Note that $J_0(0) = 1$ and $J_1(0) = 0$:

$$1 = \frac{dJ_1}{du}(0) + \lim_{u\to 0}\frac{J_1(u)}{u} = \frac{dJ_1}{du}(0) + \lim_{u\to 0}\frac{\frac{dJ_1}{du}(u)}{1}$$

$$= 2\frac{dJ_1}{du}(0) \quad \text{, or}$$

$$\lim_{u\to 0}\frac{J_1(u)}{u} = \frac{dJ_1}{du}(0) = \frac{1}{2} \quad .$$

The circular aperture (continued)

This resolves the indeterminacy:

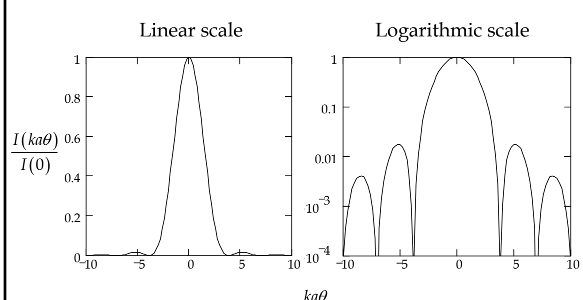
$$I_F(0) = \frac{cE_{N0}^2 A^2}{8\pi\lambda^2 r^2}\left[2\frac{1}{2}\right]^2 = \frac{cE_{N0}^2 A^2}{8\pi\lambda^2 r^2} \quad ,$$

$$\boxed{I_F(ka\theta) = I_F(0)\left[\frac{2J_1(ka\theta)}{ka\theta}\right]^2} \quad . \qquad \text{Airy pattern}$$

Because J_1 also has zeroes at finite values of $ka\theta$, $I_F(ka\theta)$ has a set of concentric rings for which the intensity is zero (*dark* rings) The first of these lies at $ka\theta_1 = 3.832$, or

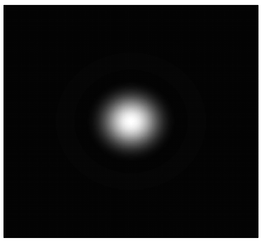
$$\theta_1 = \frac{3.832}{ka} = \frac{3.832}{\pi}\frac{\lambda}{2a} = \boxed{1.22\frac{\lambda}{D}} \quad . \qquad \text{First dark ring}$$

The Airy pattern

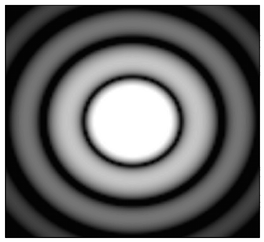


The Airy pattern (continued)

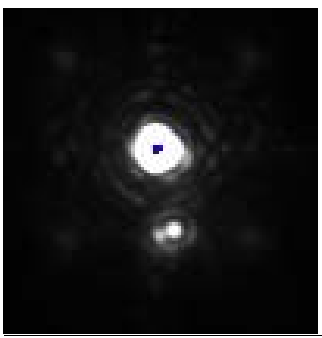
Linear scale



Logarithmic scale



The Airy pattern (continued)



A young triple-star system, T Tau, observed at $\lambda = 2.2\ \mu$m with adaptive optics on the Palomar 200-inch telescope. The brightest star has saturated the detector in the Airy disk. Note the extensive nest of concentric dark rings around it. (Linear scale.)

The opaque circular obstacle

We can handle the case of diffraction by a circular *obstacle* quite easily, using the result just obtained. For the field,

$$\begin{aligned}
E_F(q,t) &= \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \int_a^\infty dr' r' J_0\left(\frac{kqr'}{r}\right) \\
&= \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \int_0^\infty dr' r' J_0\left(\frac{kqr'}{r}\right) \\
&\quad - \frac{2\pi E_{N0} e^{i(kr-\omega t)}}{\lambda r} \int_0^a dr' r' J_0\left(\frac{kqr'}{r}\right) \\
&= E_{N0} e^{i(kz-\omega t)} - \frac{E_{N0} \pi a^2 e^{i(kr-\omega t)}}{\lambda r} \frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right) \quad ,
\end{aligned}$$

The opaque circular obstacle (continued)

and for the intensity,

$$= 2\cos kr\left(1 - z/r\right)$$
$$= 2\cos kr\left(1 - \cos\theta\right)$$
$$\cong 2\cos kr\left(\frac{\theta^2}{2}\right) = 2\cos\frac{kq^2}{2r}$$

$$I_F(q,t) = \frac{c}{8\pi} E_F E_F^*$$
$$= \frac{c}{8\pi} E_{N0}^2 \left[1 + \left(\frac{ka^2}{2r}\frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right)\right)^2 - \frac{ka^2}{r}\frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right)\boxed{\left(e^{ik(r-z)} + e^{-ik(r-z)}\right)} \right]$$
$$= \frac{c}{8\pi} E_{N0}^2 \left[1 + \left(\frac{ka^2}{2r}\frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right)\right)^2 - \frac{2ka^2}{r}\frac{r}{kaq} 2J_1\left(\frac{kaq}{r}\right)\cos\frac{kq^2}{2r} \right] .$$

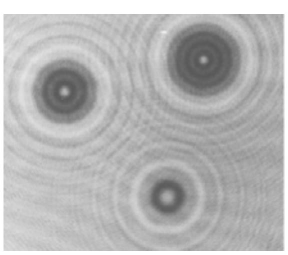
The opaque circular obstacle (continued)

This result has the curious property of not being zero inside the shadow of the obstacle. In fact, there's a sharp peak exactly in the center, with peak intensity

$$I_F(q,t) = \frac{c}{8\pi} E_{N0}^2 \left[1 + \left(\frac{ka^2}{2r}\right)^2 - \frac{2ka^2}{r} \right] .$$

And there are concentric bright and dark rings that also lie within the shadow, though generally it is much darker there than it is outside the shadow.

The opaque circular obstacle (continued)



Diffraction patterns at $\lambda = 0.635\,\mu$m, seen 5 m away from 0.09375 inch, 0.15625 inch, and 0.1875 inch diameter spheres (Ioan Feier, Horst Friedsam and Merrick Penicka, Argonne National Laboratory).

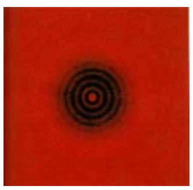
Poisson's spot

Not all effects named after famous physicists are meant to honor their namesakes.

- In 1818, the French Academy, led by neo-Newtonians like Laplace, Biot and Poisson, offered a prize for the best work on the theme of diffraction, expecting that the result would be a definitive refutation of the wave theory of light.
- Fresnel, supported by Ampère and Arago, offered a paper in which he developed the scalar theory of diffraction in much the same way we did, based on the wave theory.
- During Fresnel's talk, Poisson pointed out that one of the consequences of Fresnel's theory was the intensity peak in the center of circular shadows that we just found.

Poisson's spot (continued)

- Poisson did this, of course, because he thought such a result was ridiculous; he meant it as a fatal objection to Fresnel's theory.
- But right after the talk, Arago went into his lab, observed the intensity peak and concentric rings in the shadow directly, and proceeded to demonstrate it to the judges.
- Thus Fresnel was awarded the prize, the corpuscular theory of light stood refuted (until Einstein and Planck came along), and the intensity peak has been known ever since as **Poisson's spot**.



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