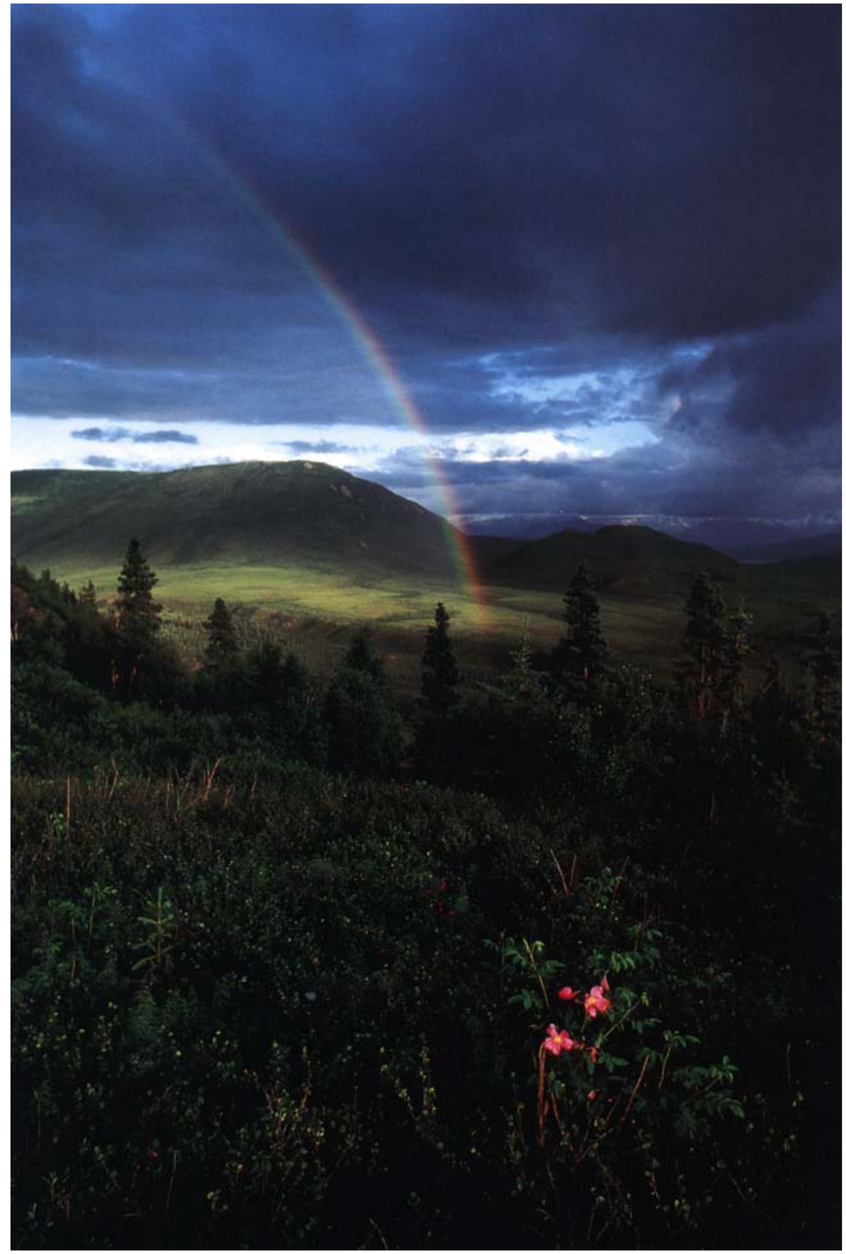
Today in Physics 218: rainbows

- The facts about rainbows, and the short explanation of all the facts
- Brief survey of the history of the study of rainbows
- The geometrical optics of raindrops
- Dispersion and the color of rainbows
- Brewster's angle and the polarization of rainbows



Rainbow and wild rose, Denali National Park. Photograph by Galen Rowell.

The facts about rainbows

1. Rainbows appear as circles or circular arcs, centered on the point on the sky exactly opposite the sun (the **antisolar point**).
   By "point," we mean direction. The Sun is far enough away to consider sunlight to be an incoherent superposition of plane waves, with all their $\boldsymbol{k}$s parallel.
2. One rainbow – the brightest (**primary**) one – lies in the direction 42° away from the antisolar point, or 138° away from the direction of the sunlight. One often sees another (**secondary**) rainbow 50° away from the antisolar point, or 130° from the sunlight.
   The sky is noticeably darker between these rainbows, compared to other directions, a phenomenon known as **Alexander's dark band.**

360° double rainbow over the Na Pali coast, Kauai. Photograph by Galen Rowell.

The facts about rainbows (continued)

3. In most cases, rainbows display a dispersed spectrum, which of course is their most striking feature. The primary bow is red on the outside (larger angles from the antisolar point) and blue on the inside, and the secondary bow has its colors the other way around.
4. Rainbows are very strongly, linearly polarized, with $\boldsymbol{E}$ tangent to the rainbow's arc (i.e. perpendicular to the plane of incidence).
5. The colors tend to be more distinct, or purer, nearer to the horizon than they are at the top of the bow.

Rainbow over the Potala Palace. Photograph by Galen Rowell.

The facts about rainbows (continued)

6. One often sees additional colored bands on the *inside* (smaller angles from the antisolar point) of the primary rainbow, in addition to the spectrum from red to violet that comprises the primary bow itself. The colors of the extra bands aren't very distinct, and are somewhat stronger near the top of the bow than they are closer to the horizon.
   The extra bands are called **supernumerary arcs**.
   In rare cases, supernumerary arcs also appear on the *outside* of the secondary rainbow, and are also brighter at the top of the bow than the bottom.

Double rainbow and the VLA, photographed in polarized light. Note the contrast of the dark band, and the variation in brightness of the supernumerary arcs.

The facts about rainbows (continued)

Why do rainbows look like that? The quick answer:

- Facts 1 and 2 (angle of primary and secondary, and the dark band) are consequences of the spherical shape of raindrops, and of Snell's law. The primary rainbow is reflected internally once within the raindrops; the secondary, twice.
- Fact 3 (spectrum) is the result of the dispersion of water at visible wavelengths.
- Fact 4 (polarization) is a Brewster effect: the first internal reflection takes place close to the Brewster angle.
- Facts 5 and 6 (purity of spectra and supernumeraries) are outcomes of diffraction of light by raindrops.

A brief survey of the history of the study of rainbows

Descriptions of rainbows are as old as literature itself. The first description we have of one of their subtle features – the dark band – is attributed to Alexander of Aphrodisias (*c.* AD 200).

- The first recorded demonstration that spherical drops of water produce dispersed spectra at the rainbow angles was done in 1304 by Theodoric of Freiburg. Theodoric used a spherical glass vessel and a pencil beam of sunlight for his demonstration.
- The first theoretical descriptions of rainbows were those by René Descartes (who used ray tracing) and Baruch Spinoza (who used Cartesian analytical geometry), both in the 1620s.

A brief survey of the history of the study of rainbows (continued)

- Around 1666, Isaac Newton improved upon Spinoza's calculation, using calculus and analytical geometry, and verified the calculations with the most accurate measurements up to that time.
- Thomas Young – he of the double-slit experiment – first recognized the supernumerary arcs as the effect of diffraction, in a paper published in 1803, though his calculation of the supernumerary positions didn't agree very well with observations.
- George Airy – he of the circular-aperture diffraction pattern, and the transmission and reflection of plane-parallel dielectrics – improved upon Young's suggestion by using Fresnel's diffraction theory to calculate the positions and strength of the supernumerary arcs (1838).

A brief survey of the history of the study of rainbows (continued)

- It is possible, though very time-consuming and difficult, to solve the electrodynamic scattering problem exactly (and in full vector regalia) for light incident on spherical dispersive dielectrics. This was first done in 1908 by Mie and Debye. The method they used, and the results they obtained, are usually referred to as "Mie theory," and are encountered wherever there are spherical scatter-ers.
- Advanced references for Airy and Mie theory:
  H.C. van de Hulst, *Light scattering by small particles* (Dover, 1981)
  Bohren, G.F., and Huffman, *Absorption and scattering of light by small particles* (Wiley, 1983)

The geometrical optics of raindrops

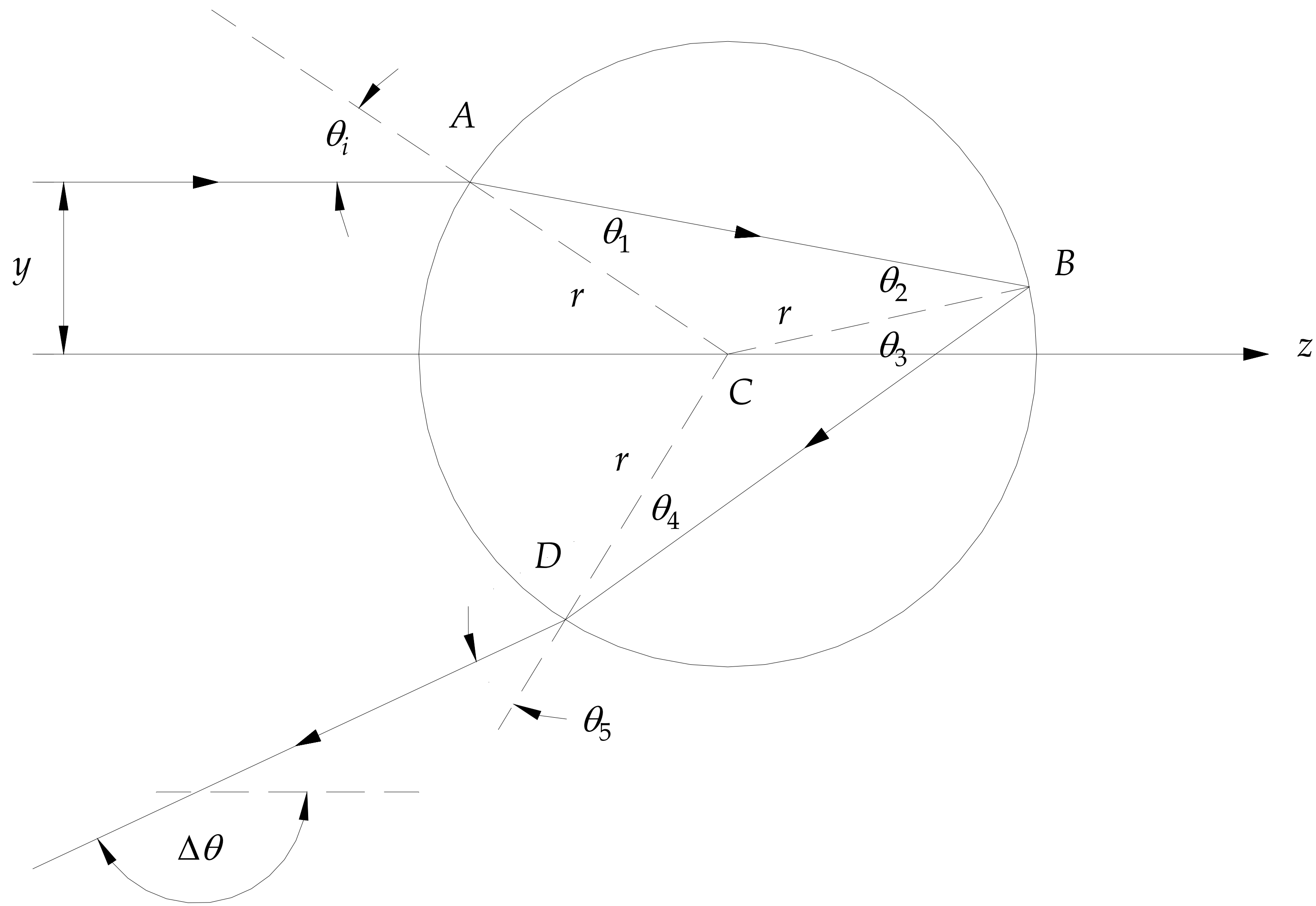
Consider a plane electromagnetic wave incident on a spherical drop of water $\left(n \cong 4/3\right)$. In particular, let's follow a small part of the wavefront incident a distance y from the center of the drop, as it refracts into the drop, reflects once internally, and emerges again. The direction of the refracted wavenumber $\boldsymbol{k}_1$ is given by Snell's law,

$$\sin\theta_i = n\sin\theta_1 \quad .$$

where the angles as usual are measured with respect to the surface normal.

- Note that all the surface normals are radii of the sphere, and that all the triangles in the figure are isosceles.

The geometrical optics of raindrops (continued)



The geometrical optics of raindrops (continued)

- In an isosceles triangle the angles opposite the congruent sides are congruent, so

$$\theta_1 = \theta_2$$

  and

$$\theta_3 = \theta_4$$

- The internal reflection follows the mirror-reflection rule, so

$$\theta_2 = \theta_3$$

  This makes *all* of the reflection/refraction angles within the drop congruent.

- Thus

$$\theta_5 = \arcsin\left(n \sin \theta_4\right) = \arcsin\left(n \sin \theta_1\right) = \theta_i$$

Let's call the angles $\theta \left(= \theta_i = \theta_5\right)$ and $\theta' \left(= \theta_1 = \theta_2 = \theta_3 = \theta_4\right)$ henceforth.

The geometrical optics of raindrops (continued)

The reason that rainbows are bright at some particular angle is that there is a minimum (or maximum) in the scattering angle, $\Delta\theta(y)$, so that many rays with different y come out at nearly the same $\Delta\theta$, near this extremum.

- To calculate this extremum, let's place the origin of coordinates at the center of the sphere, so that the boundary of the sphere in the plane of incidence is described by $y^2 + z^2 = r^2$, and the illuminated side by $z = -\sqrt{r^2 - y^2}$. Then,

$$\frac{dz}{dy} = \frac{y}{\sqrt{r^2 - y^2}} = \tan\theta \quad .$$

The geometrical optics of raindrops (continued)

- From this we have

$$\sin\theta = \frac{y}{\sqrt{\left(r^2 - y^2\right) + y^2}} = \frac{y}{r} \quad ,$$

$$\cos\theta = -\frac{\sqrt{r^2 - y^2}}{\sqrt{\left(r^2 - y^2\right) + y^2}} = -\sqrt{1 - \frac{y^2}{r^2}} \quad .$$

- From Snell's Law we have

$$\sin\theta' = \frac{1}{n}\sin\theta = \frac{y}{nr} \quad .$$

The geometrical optics of raindrops (continued)

- This is all we need to know, to calculate the scattering angle in terms of y: at point A, the light is deflected by the angle $\theta - \theta'$; at point B, another $\pi - 2\theta'$; and at point D, another $\theta - \theta'$. Thus

$$\begin{aligned}\Delta\theta &= 2\theta - 4\theta' + \pi \\ &= 2\arcsin\left(\frac{y}{r}\right) - 4\arcsin\left(\frac{y}{nr}\right) + \pi \quad .\end{aligned}$$

- We can find the extrema in $\Delta\theta$ by the usual method:

$$\frac{d}{dy}\Delta\theta = \frac{2}{\sqrt{r^2 - y^2}} - \frac{4}{\sqrt{n^2r^2 - y^2}} = 0 \quad \text{at } y = y_0 ,$$

The geometrical optics of raindrops (continued)

or $$r^2 - y_0^2 = \frac{1}{4}\left(n^2 r^2 - y_0^2\right) \quad , \quad \text{and}$$

$$y_0 = \frac{r}{3}\sqrt{12 - 3n^2} \quad .$$

- Thus no real extremum exists for $n > 2$. If water had an index greater than 2, there would be no rainbows.
- What kind of extremum is this? Test the second derivative:

$$\frac{d^2}{dy^2}\Delta\theta = \frac{2y}{\left(r^2 - y^2\right)^{3/2}} - \frac{4y}{\left(n^2 r^2 - y^2\right)^{3/2}}$$

The geometrical optics of raindrops (continued)

- Evaluate at the extremum:

$$\frac{d^2}{dy^2}\Delta\theta(y_0) = \frac{\frac{2r}{3}\sqrt{12-3n^2}}{\left(r^2 - \frac{r^2}{9}\left[12-3n^2\right]\right)^{3/2}} - \frac{\frac{4r}{3}\sqrt{12-3n^2}}{\left(n^2r^2 - \frac{r^2}{9}\left[12-3n^2\right]\right)^{3/2}}$$

$$= \frac{2\sqrt{12-3n^2}}{3r^2}\left[\frac{1}{\left(\frac{n^2-1}{3}\right)^{3/2}} - \frac{2}{\left(\frac{4n^2-4}{3}\right)^{3/2}}\right]$$

The geometrical optics of raindrops (continued)

- Thus,

$$\frac{d^2}{dy^2}\Delta\theta\left(y_0\right) = \frac{2\sqrt{12-3n^2}}{3r^2}\left[\left(\frac{3}{n^2-1}\right)^{3/2} - \frac{1}{4}\left(\frac{3}{n^2-1}\right)^{3/2}\right]$$

$$= \frac{\sqrt{12-3n^2}}{2r^2}\left(\frac{3}{n^2-1}\right)^{3/2} > 0 \quad ,$$

so if the extremum exists, it's a minimum.

- Putting the numbers in, we get

$$\Delta\theta_{\min} = 2\arcsin\left(\frac{y_0}{r}\right) - 4\arcsin\left(\frac{y_0}{nr}\right) + \pi$$

$$= 2\arcsin\left(\frac{1}{3}\sqrt{12-3n^2}\right) - 4\arcsin\left(\frac{1}{3n}\sqrt{12-3n^2}\right) + \pi$$

The geometrical optics of raindrops (continued)

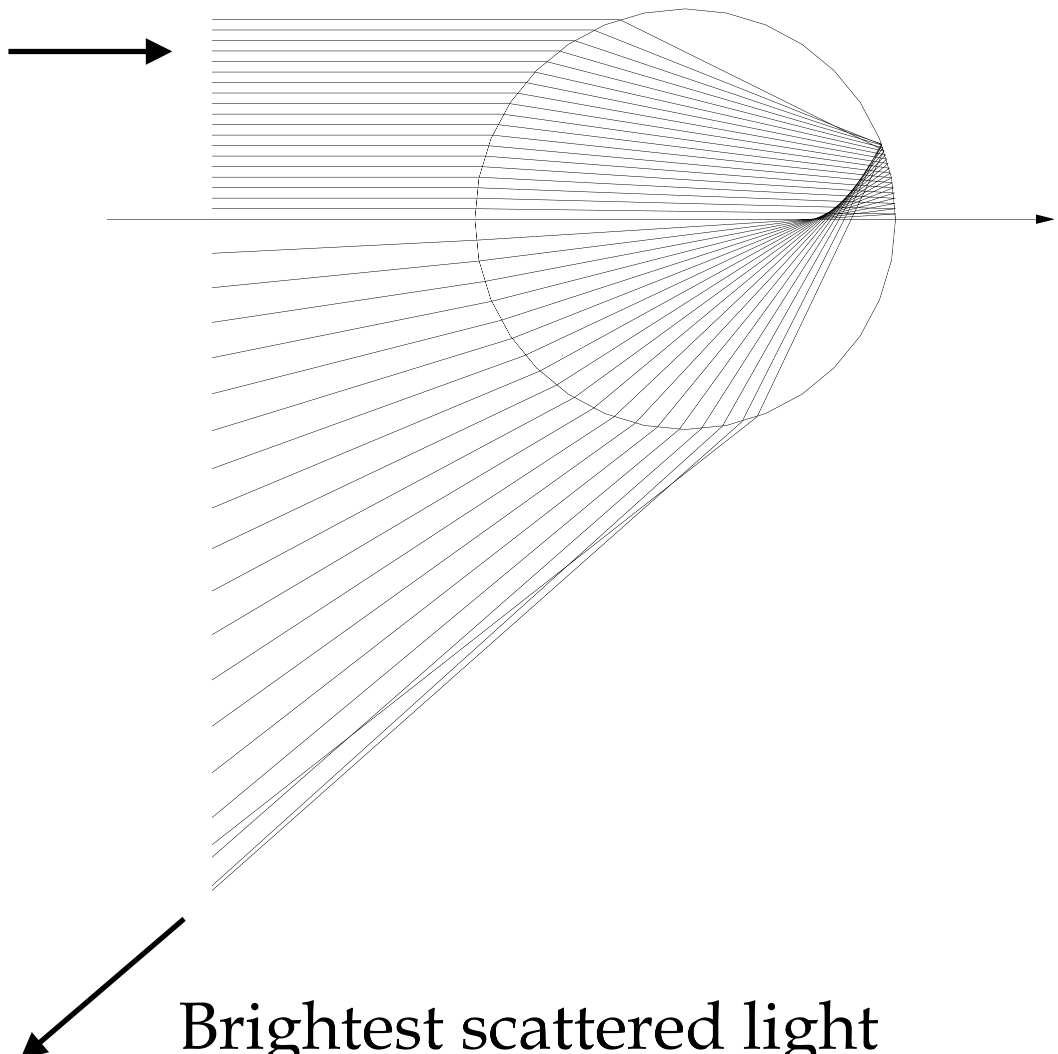
$$\Delta\theta_{\text{min}} = 2\arcsin\left(\frac{1}{3}\sqrt{12-3\left(\frac{4}{3}\right)^2}\right) - 4\arcsin\left(\frac{1}{4}\sqrt{12-3\left(\frac{4}{3}\right)^2}\right) + \pi$$

$$= 2.4080 \text{ radians} = 137.97^\circ \quad .$$

That is: 42.03° from the antisolar direction, for $n = 4/3$.

In optics, bright features formed from such extrema of reflection or refraction angles are called **caustics**.

The geometrical optics of raindrops (continued)



Brightest scattered light

The geometrical optics of raindrops (continued)

Once can calculate similarly the position of the secondary rainbow, and find out whether it's a maximum or minimum of scattering angle. You will do this on Homework #7; it will turn out that the secondary rainbow is a *maximum* of $\Delta\theta$, and that $\Delta\theta_{\max} = 130^\circ$.

- Note that these results are in excellent accord with observations.
- Note also that no light is scattered into the range $\Delta\theta = 130^\circ - 138^\circ$:this neatly accounts for Alexander's dark band.

Dispersion in water, and the colors of the rainbow

The refractive index of water varies, owing to dispersion: it's about 1.330 at λ = 0.7 μm (red) and 1.342 at 0.4 μm (violet) – so the peaks in the intensity of scattered light of shorter wavelength occur at slightly larger $\Delta\theta$, and the scattered light covers the range $\Delta\theta = 137.5° - 139.2°$.

- The angular spread of sunlight – until now considered to be zero – is actually about 0.5° (the angular diameter of the Sun). This is smaller than the spread of scattering angle for plane-wave light (1.73°), so the colors *can* be expected to be seen distinctly.
- Thus the primary rainbow should be about 2.23° in width – in excellent accord with observations.

Polarization of rainbows

For light incident at $y = y_0$ (the "rainbow ray"), the internal reflection takes place at incidence angle

$$\theta' = \arcsin\left(\frac{y_0}{nr}\right) = \arcsin\left(\frac{1}{3n}\sqrt{12 - 3n^2}\right) = 40.3^\circ \quad .$$

This is pretty close to the Brewster angle:

$$\theta_B = \arctan\left(\frac{1}{n}\right) = 36.9^\circ \quad .$$

So light with ***E*** in the plane of incidence will not reflect very well. Since rainbows need this internal reflection, it follows that they should be polarized strongly, perpendicular to the plane of incidence (i.e. tangent to the rainbow's arc).

- So far we've explained facts 1-4!