Today in Physics 218: diffraction from raindrops

- ☐ Supernumerary arcs
- ☐ Caustics and diffraction
- ☐ Airy's theory of the rainbow and the supernumerary arcs

Photo by C.R. Nave, Georgia State U.

Supernumerary arcs

Curiously, no detailed description of our Facts 5 and 6 – the purity of colors lower in the rainbow, and the presence of supernumerary arcs – predates the late eighteenth century.

- ❑ Newton, for example, doesn't mention them, though it's clear that the color of the brightest supernumerary influenced his description of the color (violet) of the blue edge of the rainbow.
- ❑ On a good day, one can see three or four supernumerary arcs near the top of a bow. They're somewhat easier to see by eye than they are to photograph.

Mikolaj and Pawel Sawicki

Photograph by Mikolaj and Pawel Sawicki .

Photograph by Mikolaj and Pawel Sawicki (enhanced from the previous one) .

Photograph by Mikolaj and Pawel Sawicki .

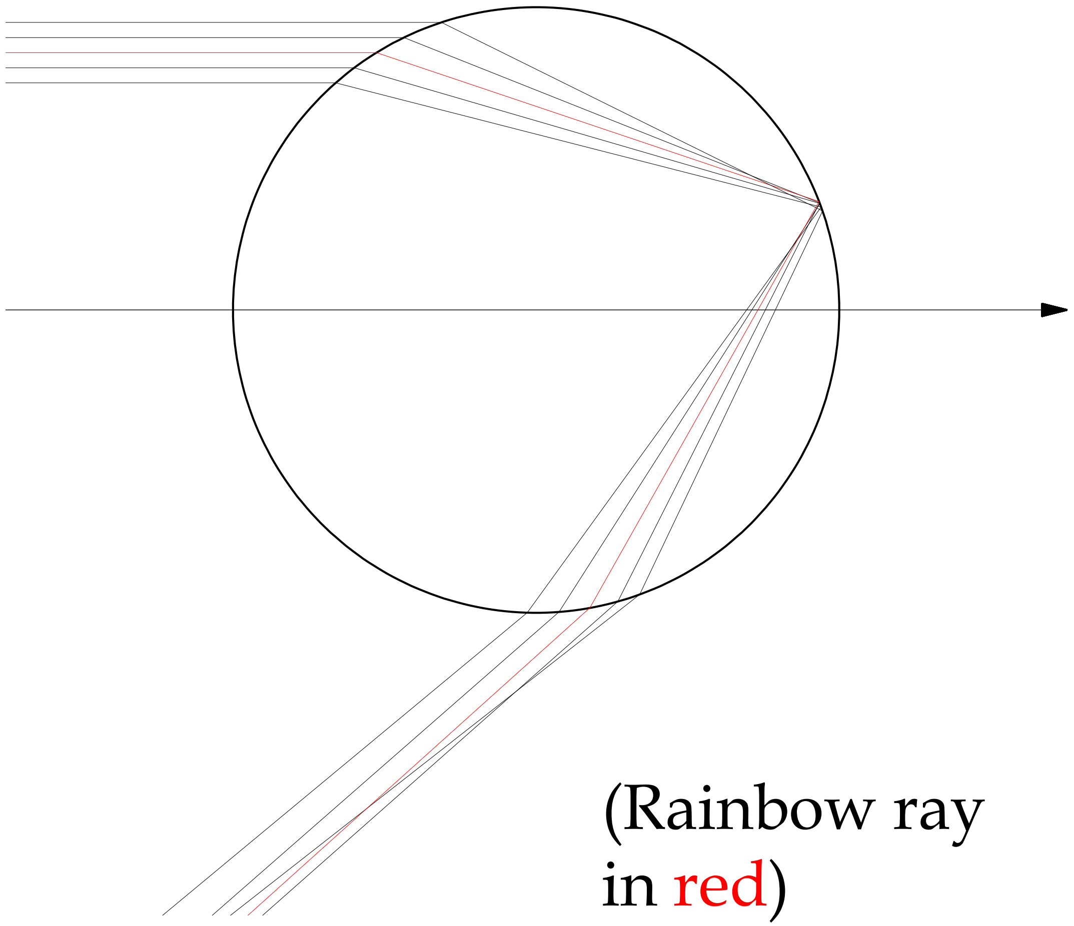
Caustics and diffraction

Airy was impressed with how beams of light resisted having sharp edges imparted to them.

- ☐ By his time, the effects of straight edges was well known, and he himself described the effects of the edges of circular bundles of rays, as we have seen.
- ☐ He was particularly interested in caustics, in which the “edge” wasn’t provided by an opaque screen. He was aware of the explanation of rainbows as caustics (by Descartes, Spinoza and Newton), and observed that the explanation of supernumeraries by Young was inaccurate, so he decided to apply the Fresnel theory of (far-field) diffraction to raindrops.

Caustics and diffraction (continued)

The calculation is made difficult by the geometry of rainbow production. Wavefronts of incident light (the surface connecting corresponding peaks of ***E*** amplitude) are planes, but those of the scattered light are not, owing to the different path lengths through the drop taken by different waves.



(Rainbow ray in red)

Airy's theory of the rainbow and the supernumerary arcs

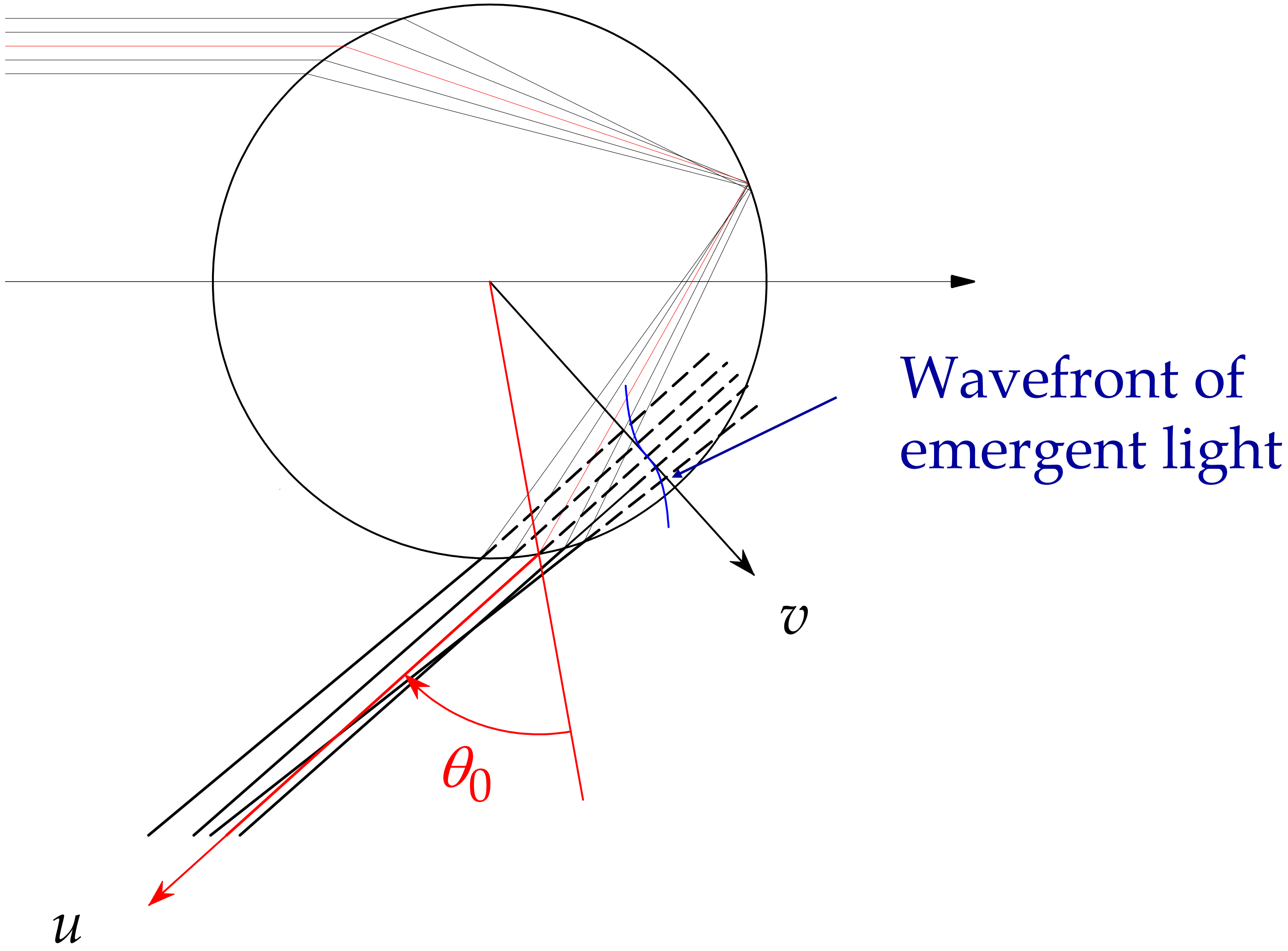
To explain the supernumeraries, Airy sought to use the far-field diffraction integral:

$$E_F\left(k_x,k_y,t\right)=\frac{e^{ikR}}{\lambda R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_N(x',y',t)e^{-i\left(k_x x'+k_y y'\right)}dx'dy' \quad .$$

He thus needed to know E_N, in the vicinity of the rainbow ray, accounting for the departure of the emergent wavefronts from planar shape.

- ❑ To express this, he constructed a coordinate system along the rainbow ray: the *u* axis along the rainbow wavenumber, and the *v* axis perpendicular to the rainbow wavenumber, outward from the center or the drop (see below).

Airy's theory of the rainbow and the supernumerary arcs (continued)



Airy's theory of the rainbow and the supernumerary arcs (continued)

- ☐ Then the field *along a plane* can be obtained, by getting the phase difference between the plane and the curved wavefront from the *distance* between them:

  $$E_N = E_N\left(v,t\right) = E_0 e^{-i\omega t} e^{iku(v)} \quad ,$$

  whence, to borrow a result from this week's team homework problem,

  $$E_F = \frac{e^{ikR}}{\lambda R} \int E_N\left(v,t\right) e^{ik_v v} dv \quad .$$

  It remains "simply" to find *u(v)*. For this we can invoke two of the results we obtained last time.

Airy’s theory of the rainbow and the supernumerary arcs (continued)

$$\frac{d}{dy}\Delta\theta = \frac{2}{\sqrt{r^2 - y^2}} - \frac{4}{\sqrt{n^2 r^2 - y^2}}$$

$$\frac{d^2}{dy^2}\Delta\theta = \frac{2y}{\left(r^2 - y^2\right)^{3/2}} - \frac{4y}{\left(n^2 r^2 - y^2\right)^{3/2}} \quad ,$$

or, using $y = r\sin\theta, dy = r\cos\theta d\theta,$

$$\frac{d}{d\theta}\Delta\theta = 2 - \frac{4\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

$$\frac{d^2}{dy^2}\Delta\theta = -\frac{4\sin\theta}{\sqrt{n^2 - \sin^2\theta}} + \frac{4\sin\theta\cos^2\theta}{\left(n^2 - \sin^2\theta\right)^{3/2}} \quad .$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

- ☐ Furthermore,

$$\frac{d^2}{dy^2}\Delta\theta = \frac{-4n^2\sin\theta + 4\sin^3\theta + 4\sin\theta\cos^2\theta}{\left(n^2 - \sin^2\theta\right)^{3/2}}$$

$$= \frac{4\sin\theta\left(1-n^2\right)}{\left(n^2 - \sin^2\theta\right)^{3/2}} \quad ,$$

which, for the rainbow ray, at

$$y_0 = \frac{r}{3}\sqrt{12-3n^2} \quad \Leftrightarrow \quad \sin^2\theta_0 = \frac{y_0^2}{r^2} = \frac{4-n^2}{3} \quad ,$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

we get

$$\frac{d^2}{dy^2}\Delta\theta = \frac{4\sin\theta_0\left(1-4+3\sin^2\theta_0\right)}{\left(4-3\sin^2\theta_0-\sin^2\theta_0\right)^{3/2}} = -\frac{12\sin\theta_0\cos^2\theta_0}{8\cos^3\theta_0}$$

$$= -\frac{3}{2}\tan\theta_0 \quad .$$

- ❑ Airy restricted his attention to the vicinity of the rainbow ray, and expanded the scattering angle in a Taylor series about the rainbow ray angle:

Airy's theory of the rainbow and the supernumerary arcs (continued)

= 0!

$$\Delta\theta(\theta) = \Delta\theta(\theta_0) + \cancel{\left[\frac{d}{d\theta}\Delta\theta\right]_{\theta=\theta_0}(\theta-\theta_0)}$$

$$+\frac{1}{2}\left[\frac{d^2}{d\theta^2}\Delta\theta\right]_{\theta=\theta_0}(\theta-\theta_0)^2 + \ldots$$

$$\cong \Delta\theta(\theta_0) + \frac{1}{2}\left[\frac{d^2}{d\theta^2}\Delta\theta\right]_{\theta=\theta_0}(\theta-\theta_0)^2$$

$$\Delta\theta(\theta) - \Delta\theta(\theta_0) = -\frac{3}{4}\tan\theta_0(\theta-\theta_0)^2 = \frac{du}{dv}$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

- ☐ Now, which $\theta - \theta_0$ corresponds to which v? Note that the geometry along v for the emergent light is the same as the geometry along y for the incident light:

$$v = r\sin\theta - r\sin\theta_0 \quad .$$

- ☐ Let $\theta = \theta_0 + \delta, \delta \ll 1$:

$$\begin{aligned} v &= r\left(\sin\left[\theta_0 + \delta\right] - \sin\theta_0\right) \\ &= r\left(\sin\theta_0\cos\delta - \cos\theta_0\sin\delta - \sin\theta_0\right) \\ &\cong r\left(\sin\theta_0 - \delta\cos\theta_0 - \sin\theta_0\right) \\ &= -r\delta\cos\theta_0 = -r\left(\theta - \theta_0\right)\cos\theta_0 \quad , \end{aligned}$$

or

$$\theta - \theta_0 = -\frac{v}{r\cos\theta_0} \quad .$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

- ☐ Thus

$$\frac{du}{dv} = -\frac{3}{4}\frac{\tan\theta_0}{r^2\cos^2\theta_0}v^2 \quad ,$$

which can be integrated trivially (noting that $u = 0$ at $v = 0$ by definition):

$$u = -\frac{\tan\theta_0}{4r^2\cos^2\theta_0}v^3$$

$$= -\frac{hv^3}{3r^2} \quad , \quad h \equiv \frac{3\tan\theta_0}{4\cos^2\theta_0} = 4.98 \text{ for } n = 4/3 .$$

- ☐ Finally (or at least semi-finally),

$$E_N\left(v,t\right) = E_0 e^{-i\omega t} e^{-ikhv^3/3r^2} \quad .$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

- So the far field is

$$E_F\left(\Delta\theta,t\right)=\frac{e^{ikR}}{\lambda R}\int\limits_{\text{drop}} E_0 e^{-i\omega t}e^{-ikhv^3/3r^2}e^{ik_v v}dv$$

$$=\frac{E_0 e^{ikR-i\omega t}}{\lambda R}\int\limits_{\text{drop}} e^{-ikhv^3/3r^2}e^{ik\left(\Delta\theta-\Delta\theta[\theta_0]\right)v}dv \quad ,$$

and the intensity is

$$I=\frac{c}{8\pi}E_F E_F^*$$

$$=\frac{cE_0^2}{8\pi\lambda^2R^2}\left(\int\limits_{\text{drop}}\cos\left[k\left(\Delta\theta-\Delta\theta\left[\theta_0\right]\right)v-k\frac{hv^3}{3r^2}\right]dv\right)^2 \quad .$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

- ☐ Now make a bunch of substitutions:

$$w = v/\ell\,, \quad dv = \ell dw$$

$$\ell = \left(\frac{3\lambda r^2}{4h}\right)^{1/3}$$

$$\zeta = \frac{4\ell\left(\Delta\theta - \Delta\theta[\theta_0]\right)}{\lambda} \quad ,$$

$$\boxed{I = \frac{cE_0^2}{8\pi\lambda^2 R^2}\left(\frac{3\lambda r^2}{4h}\right)^{2/3}\left(\int_{\text{drop}} \cos\frac{\pi}{2}\left(\zeta w - w^3\right) dw\right)^2 \quad .}$$

Airy's theory of the rainbow and the supernumerary arcs (continued)

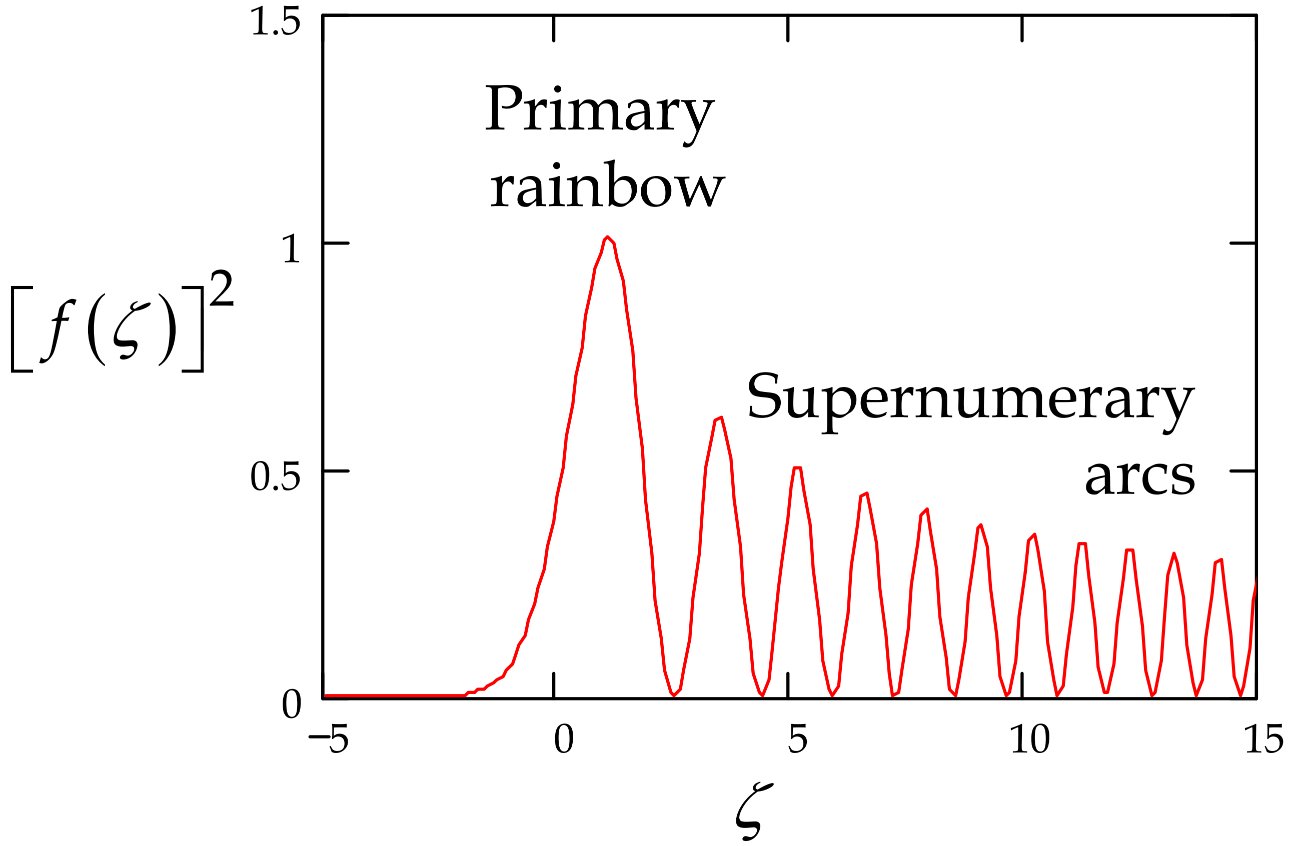
The function

$$f(\zeta) = \int_0^\infty \cos\frac{\pi}{2}\left(\zeta w - w^3\right) dw$$

turns out to be expressible as a combination of Bessel functions of order 1/3 and -1/3, but it's easy to evaluate numerically as is – the integral converges pretty fast – and that's the way Airy did it. (Without computers!)

- He called $f(\zeta)$ the "rainbow integral," but we call it the **Airy function** nowadays, in his honor.

Airy's rainbow integral



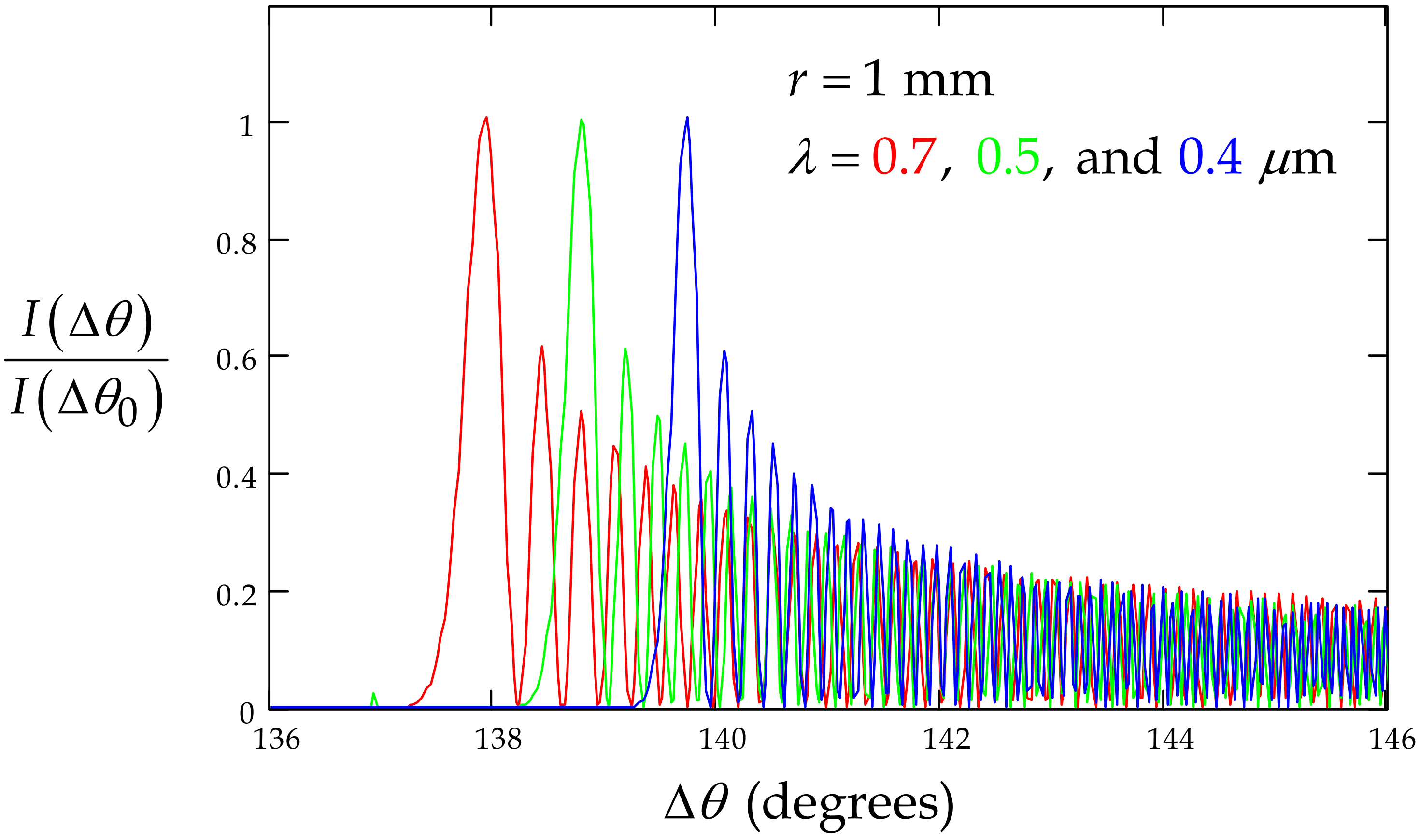
Airy's theory of the rainbow and the supernumerary arcs (continued)

This result goes a long way toward explaining facts 5 and 6.

- ❑ For large (millimeter-size) raindrops, such as those close to the ground, the primary-rainbow peak for a given wavelength is narrow compared to the angle between the peaks for blue and red light.
- ❑ The supernumeraries are very close together.
- ❑ Thus the colors of the rainbow would look very distinct, and the supernumeraries of different colors would wash each other out.

(see next page for calculation for large raindrops)

Airy's theory of the rainbow and the supernumerary arcs (continued)



Airy’s theory of the rainbow and the supernumerary arcs (continued)

- ☐ For the smaller drops – say, tenths of millimeters in size – that would be found up closer to the clouds, the angular width of the primary-rainbow peak for each wavelength is a significant fraction of the total angular width of the primary rainbow.
- ☐ The supernumeraries are more widely spaced.
- ☐ Thus the colors of the rainbow are much less distinct, and the supernumeraries can show colors, that would primarily be blue or blue-green (as near scattering angles 142° and 145° in the calculation on the next page), and lie within a few rainbow widths of the main bow. This is close to what is observed (see page 4).

Airy's theory of the rainbow and the supernumerary arcs (continued)