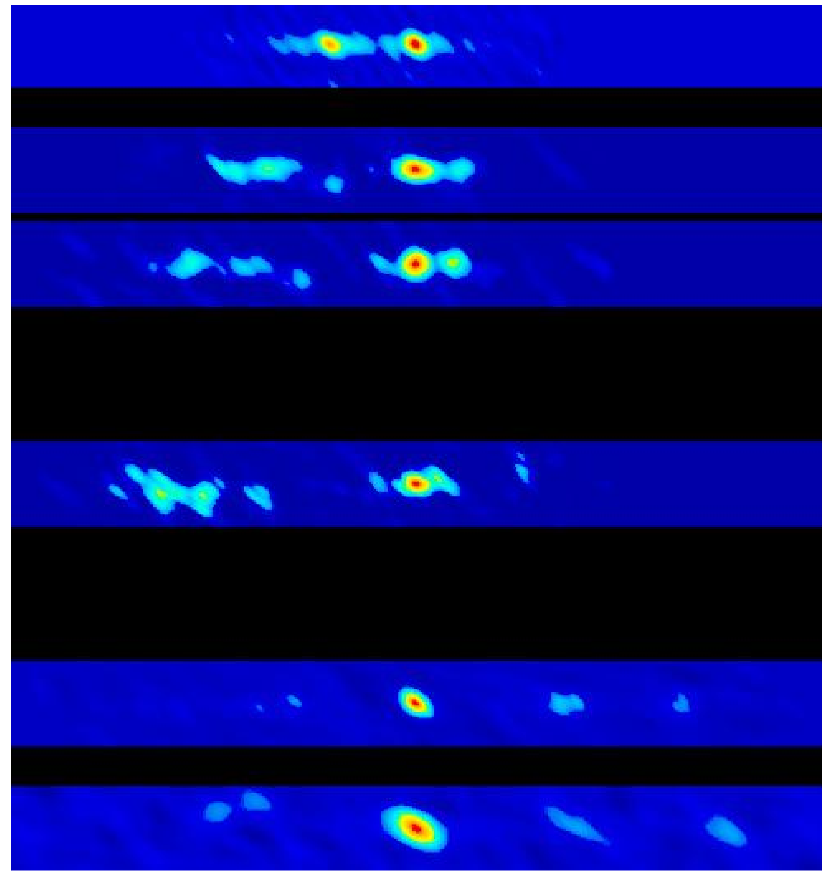
Today in Physics 218: relativity and electrodynamics

- Relativity and the four basic areas of physics
- Brief review of the basics of the special theory of relativity
- The Lorentz transformation and four-vectors
- Scalar products of four-vectors, and Lorentz invariants



Six VLBA images taken over a month, showing ejection of matter at 0.9c by the low-mass Galactic black hole GRO J1655-40 (Bob Hjellming and Mike Rupen, NRAO).

Relativity and the four basic areas of physics

Why is it that quantities familiar from the special theory of relativity, like c, $\beta = v/c$, and $\gamma = 1/\sqrt{1-(v/c)^2}$, show up so frequently in the equations of electrodynamics?

- Because, as it turns out, the special theory of relativity is already built into the Maxwell equations.
- That makes electricity and magnetism different from the other three basic branches of physics – mechanics, statistical mechanics, and quantum mechanics – all of which have to be modified (some quite drastically) in order to make them consistent with the special theory of relativity.
- We will spend the rest of the semester exploring this connection.

Review of the basics of the special theory of relativity

As you'll no doubt recall, the special theory of relativity was constructed by Albert Einstein precisely a century ago, in response to some striking failures of the prevailing understanding of electrodynamics.

- Electromagnetic radiation was considered, in analogy with vibrations in continuous media, to represent an elastic deformation of an underlying, universally-pervasive medium called the *æther.*
- The original form of the Maxwell equations applied only for an observer at rest with respect to the æther: if one were in motion with respect to the æther, the transformed form of the Maxwell equations were quite a bit more complicated.

Review of the basics of the special theory of relativity (continued)

- Among the predictions of this version of electrodynamics was that an observer could measure his or her own speed with respect to the æther, by measuring the apparent speed of light (= *c* only in the æther's rest frame).
- But Michelson, in a series of famous experiments, showed that the speed of light throughout the year (during which Earth's orbital motion modulate the speed by ±30 km/sec) exhibited no variation, to high accuracy.
- Lorentz and Fitzgerald noted that Michelson's results could result from a force, exerted by the æther on anything that moved through it, that compresses any measurement apparatus by precisely the right amount to mask the observer's motion through the æther:

Review of the basics of the special theory of relativity (continued)

$$L_{\|} = \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{L_{\|0}}{\gamma} \quad , \quad L_{\perp} = L_{\perp 0} \quad .$$

Lorentz-Fitzgerald contraction

- Lorentz went on to show that measurements of position and time of an event under this model of the æther force, made in a reference frame moving at velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ with respect to æther, such that the origin of that frame and a Rest frame coincided at $t = t' = 0$, were in error according to

$$x' = \gamma\left(x - vt\right) \quad ,$$

$$y' = y \quad , \quad z' = z \quad ,$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad .$$

Lorentz transformation
Event happens at (x,y,z,t), but is observed to happen at (x',y',z',t').

Review of the basics of the special theory of relativity (continued)

- Furthermore, showed Lorentz, under this transformation, the Maxwell equations are invariant in form; thus the moving observer would not only be fooled by measurements of position and time, but also into thinking that the Maxwell equations were the same in the moving frame as in the æther's frame.
- So far this had the virtue of involving only conventional physical concepts, such as forces, a small modification to Hooke's law, waves as displacements in an elastic medium, and the Maxwell and Galilean theories.
- It also had the disadvantage of an increasingly "ethereal" propagation medium, a force of unknown origin and nature, and much else that is *ad hoc.*

Review of the basics of the special theory of relativity (continued)

- This was all too much for Einstein. He was sure that this is far too complicated a way for nature to do its business; he was especially suspicious of the idea that the Maxwell equations, compact and symmetrical in the æther's frame, took such a hideous form in a moving frame.
- He also saw a much simpler way to do it: he proposed that
  - there is no such thing as the æther;
  - in fact, there is no such thing as a universal rest frame;
  - the rejection of "universal rest" can be accommodated by changing the theory's principles of relativity, rather than its physical principles.

Review of the basics of the special theory of relativity (continued)

All of the special theory of relativity can be derived from two postulates:

- The laws of physics apply, and have the same form, in all inertial reference frames.
- The speed of light in vacuum is **absolute**: the same for all observers in inertial reference frames, regardless of the motion of the source of light.

where an **inertial reference frame** is one in which an observer feels no inertial forces, such as one in uniform motion at constant velocity.

- Easier to define unambiguously in pairs: two inertial reference frames can move with respect to each other only at constant velocity.

Review of the basics of the special theory of relativity (continued)

Among the first results Einstein derived from the two postulates are these:

- **Length is relative, not absolute**: moving objects *appear* to be shorter along their direction of motion than they are at rest, by exactly the amount Lorentz and Fitzgerald thought they *are* shorter.
- **Time duration is relative, not absolute**: moving clocks appear to tick more slowly than they do at rest, according to

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma \Delta t_0 \quad .$$

Review of the basics of the special theory of relativity (continued)

- **Simultaneity is relative, not absolute:** two events, that appear to happen simultaneously at different places in one inertial frame, will not in general appear to be simultaneous when viewed from other inertial frames.
- **Time and position are both relative,** in exactly the way Lorentz proposed for the extent to which an observer would be fooled by æther compression:

$$x' = \gamma\left(x - vt\right) \quad ,$$
$$y' = y \quad , \quad z' = z \quad ,$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad .$$

We still call it the Lorentz transformation, though.

Review of the basics of the special theory of relativity (continued)

- **Velocities are still relative** (except for that of light), but not as they are in Galilean relativity: for one-dimensional motion, it goes like this:

$$u' = \frac{u+v}{1+\dfrac{uv}{c^2}} \quad .$$

(Note that this automatically makes the speed of light c in all inertial frames: if $u = c$, $u' = c$.)

We want now to apply these principles to the relativity of forces, fields and potentials. To do this, it will be convenient to introduce the conventional language of relativity: spacetime, four-vectors, and invariant intervals.

The Lorentz transformation and four-vectors

Remember **rotation**, a simple coordinate transformation covered (for instance) early in PHY 217?

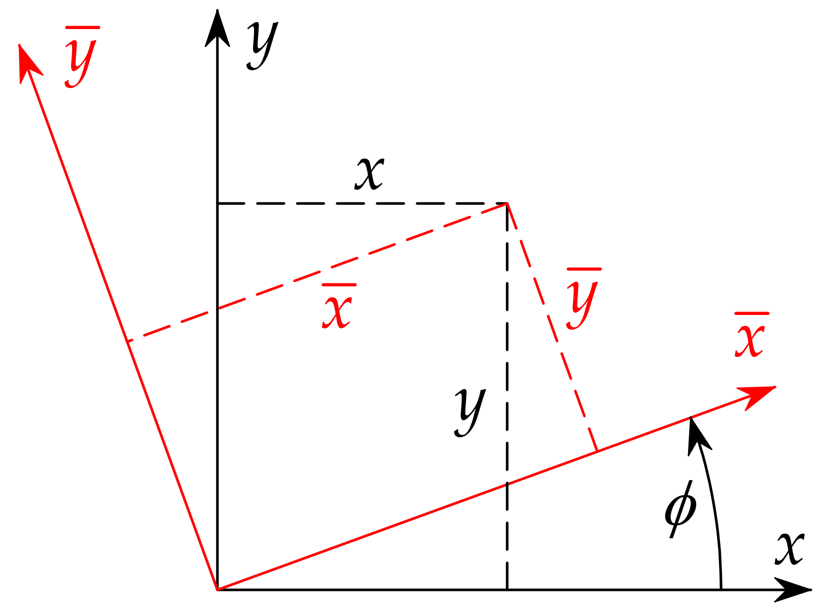
$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad ,$$

$$\text{or} \qquad \bar{x}_i = \sum_{j=1}^{3} R_{ij} x_j \quad ,$$

where $(x_1, x_2, x_3) = (x, y, z)$. As a concrete example, consider a counterclockwise rotation by an angle ϕ about the z axis, for which

The Lorentz transformation and four-vectors (continued)

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad .$$



Vectors are defined as objects that transform like the coordinates x, y, and z do under rotations:

$$\boldsymbol{A} \rightarrow \bar{\boldsymbol{A}} \quad \Leftrightarrow \quad \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j \quad .$$

The Lorentz transformation and four-vectors (continued)

Similarly, second-rank tensors are defined as those objects which transform with *two* applications of the rotation matrix:

$$\overleftrightarrow{\boldsymbol{T}} \rightarrow \overline{\overleftrightarrow{\boldsymbol{T}}} \quad \Leftrightarrow \quad \overline{T}_{ij} = \sum_{j=1}^{3} R_{ik} T_{kl} \tilde{R}_{lj} \quad ,$$

where $\tilde{R}_{lj} = R_{jl}$ is the transpose of the rotation matrix.

Can we make the Lorentz transformations look like a matrix operation? Sure. Consider two inertial frames moving relative to one another at velocity $\boldsymbol{v}$ along their x axes:

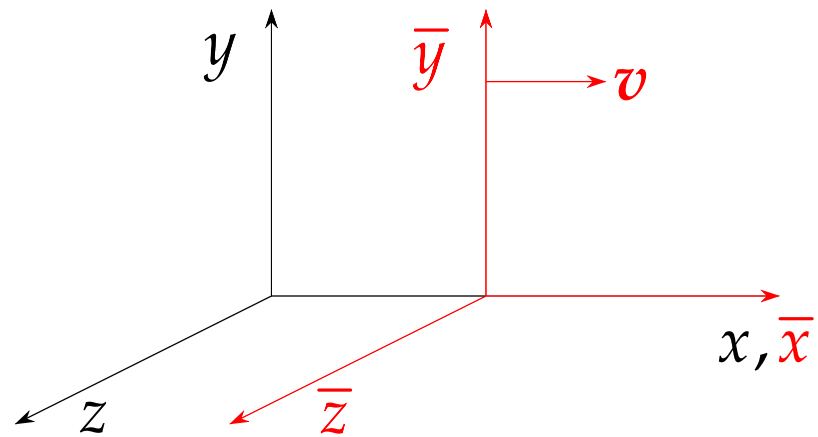
The Lorentz transformation and four-vectors (continued)

$$\bar{t} = \gamma\left(t - \frac{vx}{c^2}\right) \quad ,$$

$$\bar{x} = \gamma(x - vt) \quad ,$$

$$\bar{y} = y \quad ,$$

$$\bar{z} = z \quad .$$



Define $x^0 = ct, x^1 = x, x^2 = y,$ and $x^3 = z,$ so that

$$\bar{x}^0 = \gamma\left(x^0 - \beta x^1\right) \quad ,$$

$$\bar{x}^1 = \gamma\left(x^1 - \beta x^0\right) \quad ,$$

$$\bar{x}^2 = x^2 \quad ,$$

$$\bar{x}^3 = x^3 \quad ,$$

The Lorentz transformation and four-vectors (continued)

and we *can* write it as a matrix operation:

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad ,$$

or $$\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda_\nu^\mu x^\nu \quad ,$$

where Λ is the matrix above, for which the row and column indices are μ and ν, respectively. (For motion in other directions, some of the zeroes change to finite values.)

The Lorentz transformation and four-vectors (continued)

In analogy with rotations in 3-D, then, we can define a **four-vector** to be a four-component object that behaves the same as the coordinates under Lorentz transformation:

$$\boxed{\bar{a}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu} \quad .}$$

Contravariant four-vector

Similarly, there are higher-rank four-objects, like second-rank four-tensors:

$$\bar{t}^{\mu\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\kappa} t^{\kappa\lambda} \Lambda^{\nu}_{\lambda} \quad .$$

(Convention: Greek indices run 0-3, Latin ones 1-3.)

Scalar products of four-vectors, and Lorentz invariants

The scalar product of two vectors is of course rotationally invariant: it has the same value in any rotated coordinate system. This is because the magnitude of a vector mustn't change under rotations:

$$\bar{\mathbf{A}} \cdot \bar{\mathbf{A}} = \sum_{i=1}^{3} \bar{A}_i \bar{A}_i = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} R_{ij} A_j R_{ik} A_k$$

$$= \sum_{j=1}^{3} \sum_{k=1}^{3} \left(\sum_{i=1}^{3} R_{ij} R_{ik} \right) A_j A_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_{jk} A_j A_k$$

$$= \sum_{j=1}^{3} A_j A_j = \mathbf{A} \cdot \mathbf{A} \quad ,$$

Scalar products of four-vectors, and Lorentz invariants (continued)

that is, rotation is a **unitary** transformation, among the implications of which is

$$\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk} \quad ,$$

as you might have demonstrated once upon a time in PHY 217 (e.g. Griffiths problem 1.8). Thus,

$$\begin{aligned} \bar{\boldsymbol{A}} \cdot \bar{\boldsymbol{B}} &= \sum_{i=1}^{3} \bar{A}_i \bar{B}_i = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} R_{ij} A_j R_{ik} B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_{jk} A_j B_k \\ &= \sum_{j=1}^{3} A_j B_j = \boldsymbol{A} \cdot \boldsymbol{B} \quad . \end{aligned}$$

Scalar products of four-vectors, and Lorentz invariants (continued)

What about inner products of two four vectors? As you might expect, this operation generates an object that is invariant under Lorentz transformations. However, it turns out that the invariant has the form

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \quad .$$

To book-keep the minus sign, we introduce the **covariant** four-vector:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad ,$$

Scalar products of four-vectors, and Lorentz invariants (continued)

so that

$$\sum_{\mu=0}^{3} a_\mu b^\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \quad .$$

That this is Lorentz-invariant can be demonstrated, for reference frames in relative motion along x, as follows (Griffiths problem 12.17):

$$\sum_{\mu=0}^{3} \overline{a}_\mu \overline{b}^\mu = -\overline{a}^0 \overline{b}^0 + \overline{a}^1 \overline{b}^1 + \overline{a}^2 \overline{b}^2 + \overline{a}^3 \overline{b}^3$$

$$= -\gamma\left(a^0 - \beta a^1\right)\gamma\left(b^0 - \beta b^1\right)$$

$$+ \gamma\left(a^1 - \beta a^0\right)\gamma\left(b^1 - \beta b^0\right) + a^2 b^2 + a^3 b^3$$

Scalar products of four-vectors, and Lorentz invariants (continued)

$$\sum_{\mu=0}^{3} \bar{a}_\mu \bar{b}^\mu = -\gamma^2\left(a^0b^0 - \cancel{\beta a^1 b^0} - \cancel{\beta a^0 b^1} + \beta^2 a^1 b^1\right)$$

$$+\gamma^2\left(a^1b^1 - \cancel{\beta a^0 b^1} - \cancel{\beta a^1 b^0} + \beta^2 a^0 b^0\right) + a^2b^2 + a^3b^3$$

$$= -\gamma^2\left(a^0b^0 - \beta^2 a^0 b^0\right) + \gamma^2\left(a^1b^1 - \beta^2 a^1 b^1\right) + a^2b^2 + a^3b^3$$

$$= -\gamma^2\left(1-\beta^2\right)a^0b^0 + \gamma^2\left(1-\beta^2\right)a^1b^1 + a^2b^2 + a^3b^3$$

$$= -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3 = \sum_{\mu=0}^{3} a_\mu b^\mu$$

And we can always rotate the coordinates to line the x axis up with the relative motion, so this serves as *proof* of invariance.