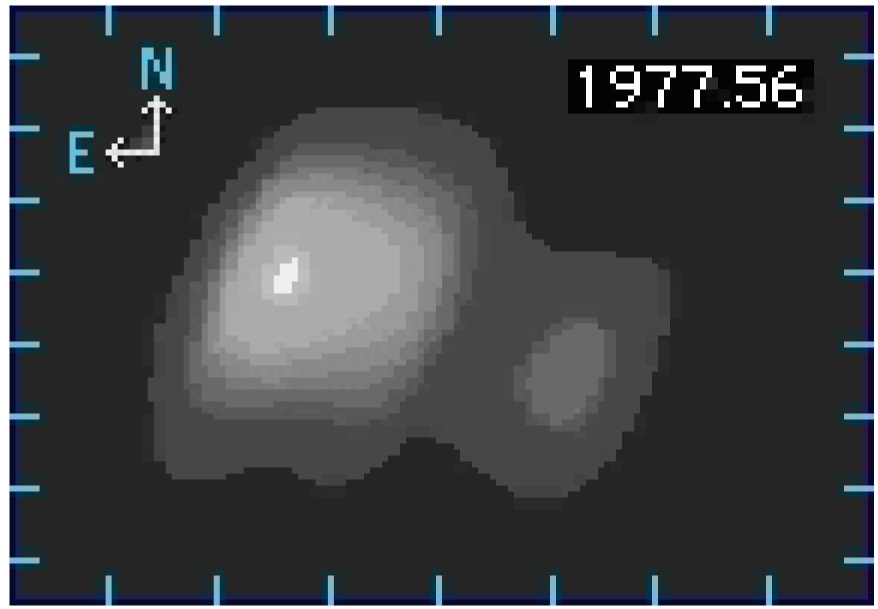
Today in Physics 218: Lorentz invariants

- The Einstein summation convention
- The Minkowski invariant interval
- Proper time and four-velocity
- Four-momentum and the relativistic energy



Movie made from VLBI radio observations of the quasar 3C273, showing the ejection of material from the active nucleus at apparent speed 7c (!) transverse to the line of sight (Tim Pearson, Caltech). See problem 12.6, on the next homework, to understand why the ejecta look like they move faster than light.

Scalar products of four-vectors, and the summation convention

Last time we introduced a bookkeeping device for scalar products, the contravariant and covariant forms of four-vectors:

$$a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad , \quad a_\mu = \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix} \quad .$$

The scalar product can be obtained either by multiplication of the vectors that represent the four-vectors:

Scalar products of four-vectors, and the summation convention (continued)

$$\begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$= -a^0 b^0 + \boldsymbol{a} \cdot \boldsymbol{b} \quad ,$$

or more compactly as

$$\sum_{\mu=0}^{3} a_\mu b^\mu = -a^0 b^0 + \boldsymbol{a} \cdot \boldsymbol{b} \quad .$$

Scalar products of four-vectors, and the summation convention (continued)

Or even more compactly. When Einstein started using four-vectors in relativity, he quickly got tired of writing all the sums, and began using the following convention: for an index that is *repeated, once covariant and once contravariant,* one assumes that the term in which it appears is to be summed over that index from zero to 3:

$$a_\mu b^\mu \quad \Leftrightarrow \quad \sum_{\mu=0}^{3} a_\mu b^\mu$$

$$a_\mu b^\nu \quad \Rightarrow \ \text{no sum}$$

$$a_\mu b_\mu \quad \Rightarrow \ \text{there must be some mistake...}$$

We will use **Einstein's summation convention** henceforth.

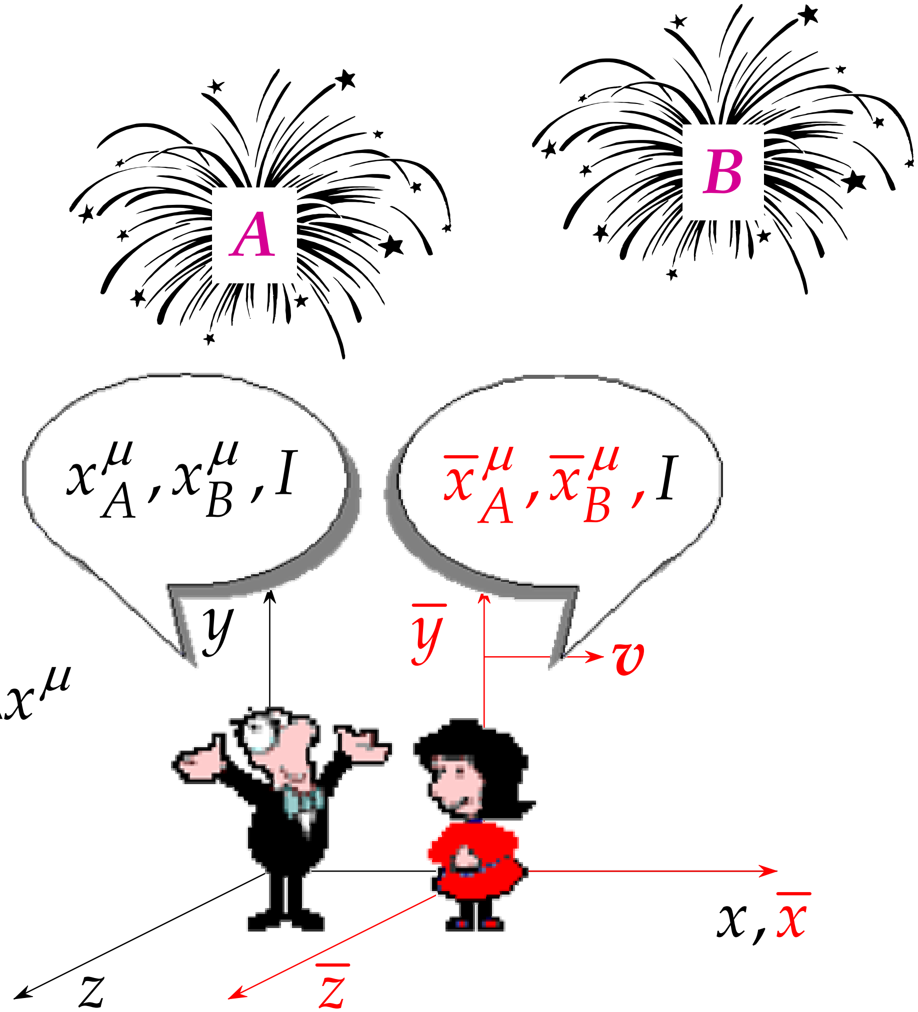
Useful scalar products: the invariant interval

Consider two events A and B, that can be seen by observers in two different inertial reference frames. The observers would give different coordinates for the events, $x_A^\mu, x_B^\mu, \bar{x}_A^\mu$, and $\bar{x}_B^\mu$, but the same value of

$$I = \left(x_{A\mu} - x_{B\mu}\right)\left(x_A^\mu - x_B^\mu\right) = \Delta x_\mu \Delta x^\mu$$

$$= -c^2 \Delta t^2 + d^2 = -c^2 \Delta \bar{t}^2 + \bar{d}^2 \quad .$$

This is called the **invariant interval.**



Useful scalar products: the invariant interval (continued)

If $I < 0$, the interval is dominated by the time-difference part, and is described as **timelike** as a result.

- Since $I < 0$, there are no two frames of reference for which $\Delta t = 0$ (because $d^2 \geq 0$),but one *can* have $d^2 = 0$, so that A and B can appear to one observer to happen at the same spatial point.
- The $I < 0$ case describes all pairs of events that are connected by **cause and effect**, since in this case the time order is always preserved: if $I < 0$ and A occurs before B in one reference frame, if occurs before B in all inertial reference frames.

Useful scalar products: the invariant interval (continued)

If $I > 0$, on the other hand, one can always find a frame in which A and B are simultaneous $(\Delta t = 0)$, as well as pairs of frames in which the events occur in different order. Such an interval is called **spacelike**.

Example (Griffiths problem 12.21):

The coordinates of events A and B are $(t_A, x_A, 0, 0)$, $(t_B, x_B, 0, 0)$. Assuming the interval between them is spacelike, find the velocity of the reference frame in which they are simultaneous.

Solution: in the new frame,

$$
\begin{aligned}
0 &= c\Delta\bar{t} = c\left(\bar{t}_A - \bar{t}_B\right) \\
&= \gamma\left(t_A - \beta x_A\right) - \gamma\left(t_B - \beta x_B\right) = \gamma\left(c\Delta t - \beta\Delta x\right) \quad .
\end{aligned}
$$

Useful scalar products: the invariant interval (continued)

Thus,

$$\Delta t = \frac{\beta}{c}\Delta x \quad , \text{ or}$$

$$v = \beta c = \boxed{\frac{\Delta t}{\Delta x}c^2} \quad .$$

- Since for spacelike separations one can find pairs of frames in which the two events occur in opposite orders, these sorts of separation cannot apply to events that are related by cause and effect.

Useful scalar products: the invariant interval (continued)

The case I **=** 0 is referred to as a **lightlike** separation, because

$$\frac{d^2}{(\Delta t)^2} = c^2$$

in all reference frames. Two events separated by a lightlike interval can be causally related only if the "cause" travels at the speed of light.

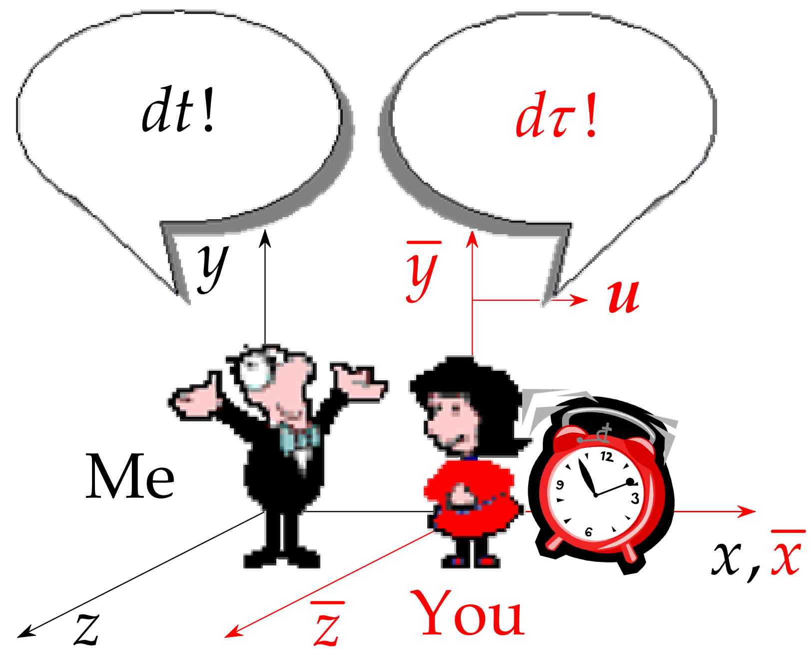
Proper time and four-velocity

The **proper time,** $d\tau$, is a time interval defined in a clock's rest frame, from the *dilated* interval observed in another frame:

$$d\tau = dt\sqrt{1-\frac{u^2}{c^2}} \quad .$$

It's invariant because there's only one frame at rest with respect to your clock. Your speed, according to me, is

$$\boldsymbol{u} = \frac{d\ell}{dt} \qquad \left\{ \begin{array}{l} \ell, t \text{ measured} \\ \text{in my frame} \end{array} \right.$$



Proper time and four-velocity (continued)

According to *your* time, though, your speed is

$$\eta = \frac{d\ell}{d\tau} = \frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u u \quad . \qquad \begin{cases} \ell \text{ measured in my frame,} \\ \tau \text{ in yours.} \end{cases}$$

It is useful to construct a four-vector velocity from this new object η :

$$\boldsymbol{\eta} = \frac{d\boldsymbol{x}}{d\tau} = \frac{\boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u \boldsymbol{u} \quad , \quad \eta^0 = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u c \quad ,$$

$$\eta^\mu = \frac{dx^\mu}{d\tau} \quad . \qquad \text{Four-velocity}$$

Proper time and four-velocity (continued)

It's a four-vector, and thus Lorentz-transforms like one:

$$\bar{\eta}^{\mu} = \Lambda^{\mu}_{\nu}\eta^{\nu} \quad ,$$

because x^{μ} is a four-vector, and the derivative with respect to proper time is as invariant as proper time.

- Contrast this with $\boldsymbol{u}$: this velocity is *not* part of a four-vector, because both ℓ and t need to be transformed. This is why the Einstein velocity addition rule you learned in freshman physics is more complicated than the Lorentz transformation.

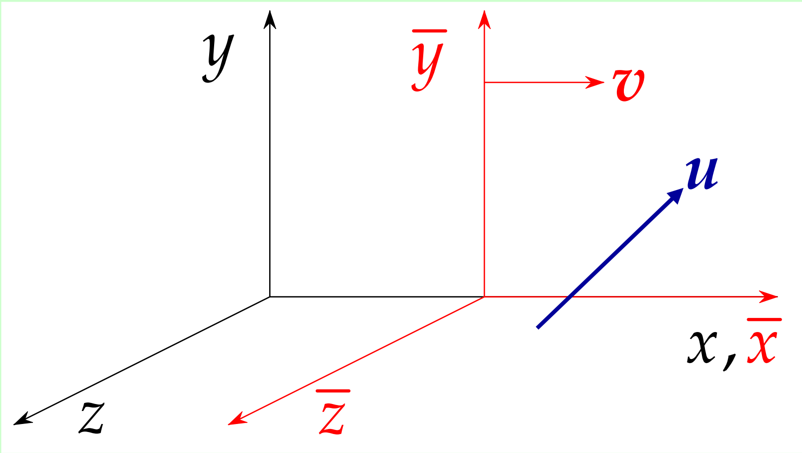
Flashback: Einstein velocity addition rule

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

$$\bar{u}_y = \frac{1}{\gamma}\frac{u_y}{1 - \dfrac{vu_x}{c^2}}$$

$$\bar{u}_z = \frac{1}{\gamma}\frac{u_z}{1 - \dfrac{vu_x}{c^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$



Four-momentum and relativistic energy

From the four-velocity, one can define the **four-momentum,** or "energy-momentum four-vector," as

$$p^0 = m\eta^0 = (m\gamma_u)c = \frac{\left(\gamma_u mc^2\right)}{c} = \frac{E}{c} \quad ,$$

$$\boldsymbol{p} = m\boldsymbol{\eta} \quad ,$$

where m is the **rest mass** of the moving body – also an invariant, because nothing has more than one rest frame.

- The invariant that can be constructed from the scalar product of the four-momentum with itself expresses relativistic energy and momentum conservation (note: the following is Griffiths problem 12.26, more or less):

Four-momentum and relativistic energy (continued)

$$p_\mu p^\mu = -\frac{E^2}{c^2} + p^2 = -\left(p^0\right)^2 + \left(\boldsymbol{p}\cdot\boldsymbol{p}\right)$$

$$= -\left(m\eta^0\right)^2 + m^2\boldsymbol{\eta}\cdot\boldsymbol{\eta} = -\frac{m^2c^2}{1-\dfrac{u^2}{c^2}} + \frac{m^2\boldsymbol{u}\cdot\boldsymbol{u}}{1-\dfrac{u^2}{c^2}}$$

$$= m^2c^2\left(-\frac{1}{1-u^2/c^2} + \frac{u^2}{1-u^2/c^2}\right) = -m^2c^2$$

$$\Rightarrow \quad E^2 - p^2c^2 = \left(mc^2\right)^2 \quad .$$

Since the right-hand side is constant, or equivalently since $\overline{p}_\mu\overline{p}^\mu = p_\mu p^\mu$, $\overline{E}^2 - \overline{p}^2c^2 = E^2 - p^2c^2$.

Four-momentum and relativistic energy (continued)

The energy-momentum invariant, and the Lorentz transformation of the four-momentum, are very useful in solving problems on the kinematics of relativistic particles.

Example (Griffiths problem 12.30): Suppose you have a collection of particles, all moving in the x direction, with energies $E_1, E_2, \ldots$ and momenta $p_1, p_2, \ldots$ Find the velocity of the **center of momentum** frame, in which the total momentum is zero.

Solution:

Let $E_T = E_1 + E_2 + \ldots$ and $p_T = p_1 + p_2 + \ldots$ Then,

$$\bar{p}_T = \gamma\left(p_T - \beta\frac{E_T}{c}\right) = 0 \quad , \text{ or } \quad v = \beta c = \boxed{\frac{c^2 p_T}{E_T}} \quad .$$

Four-momentum and relativistic energy (continued)

Example (Griffiths problem 12.33): A neutral pion, with rest mass m and relativistic momentum $p = 3mc/4$, decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the energy of each photon.

Solution: The total energy of the pion is given by

$$E^2 = p^2c^2 + \left(mc^2\right)^2 = \left(\left(\frac{3}{4}\right)^2 + 1\right)\left(mc^2\right)^2 = \frac{25}{16}\left(mc^2\right)^2 \quad , \text{ or}$$

$$E = \frac{5}{4}mc^2 \quad ,$$

and energy and momentum are conserved, so the energy and momentum of the photons (1 and 2) are given by

Four-momentum and relativistic energy (continued)

$$E_1 + E_2 = \frac{5}{4}mc^2 \quad , \quad p_1 - p_2 = \frac{3}{4}mc \quad .$$

But photons have zero rest mass $\left(\Rightarrow E = pc\right)$, so

$$E_1 + E_2 = \frac{5}{4}mc^2 \quad ,$$

$$E_1 - E_2 = \frac{3}{4}mc^2 \quad ,$$

where we've take photon 1 to travel along the pion's direction. Add these to get E_1, subtract them to get E_2 :

$$\boxed{E_1 = mc^2 \quad , \quad E_2 = \frac{1}{4}mc^2 \quad .}$$