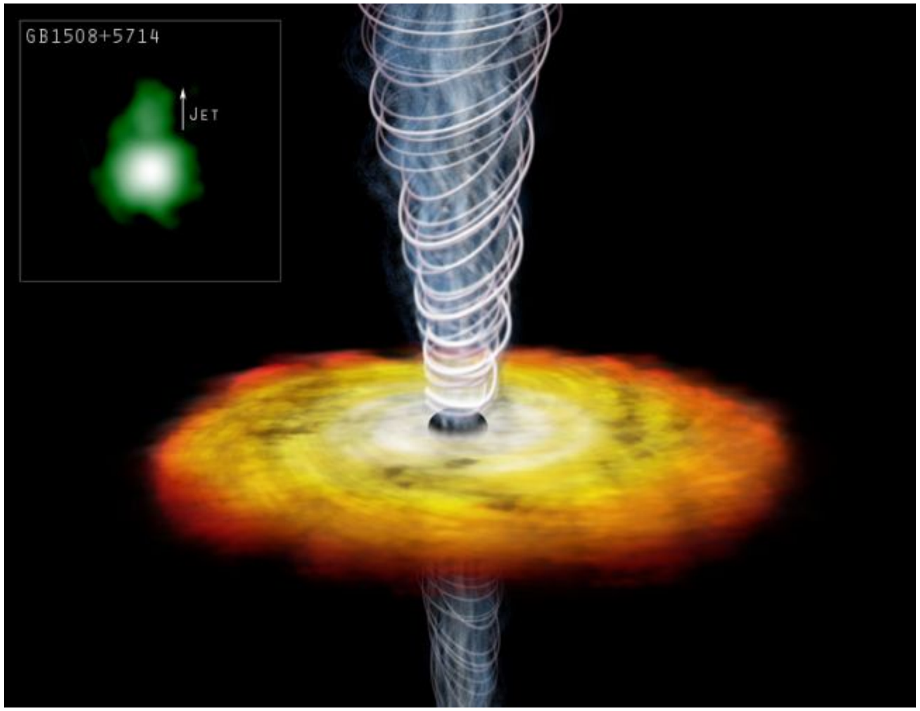
Today in Physics 218: the electromagnetic field tensor

- Relativistic transformations of ***E*** and ***B***.
- Example of the use of the transformations.
- The electromagnetic field four-tensor.



X-ray image/artist's impression of the quasar GB 1508+5714, by A. Siemiginowska/M.Weiss with the Chandra X-ray Observatory (CfA/NASA).

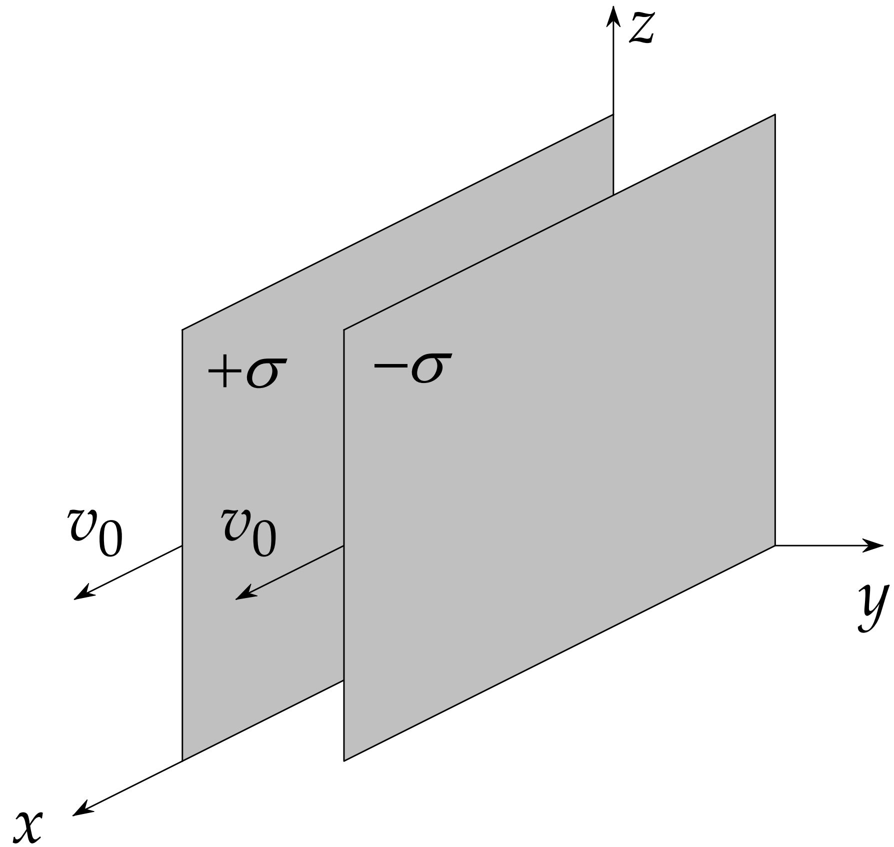
Transformation of E and B in special relativity (continued)

Last time we found, from the example of infinite charged sheets moving according to that shown at right, that field components in our frame and that of an observer moving at velocity $v\hat{\boldsymbol{x}}$ are related by

$$\overline{E}_y = \gamma\left(E_y - \beta B_z\right) \quad ,$$

$$\overline{B}_z = \gamma\left(B_z - \beta E_y\right) \quad .$$

Now orient the planes in other directions, to isolate other field components.



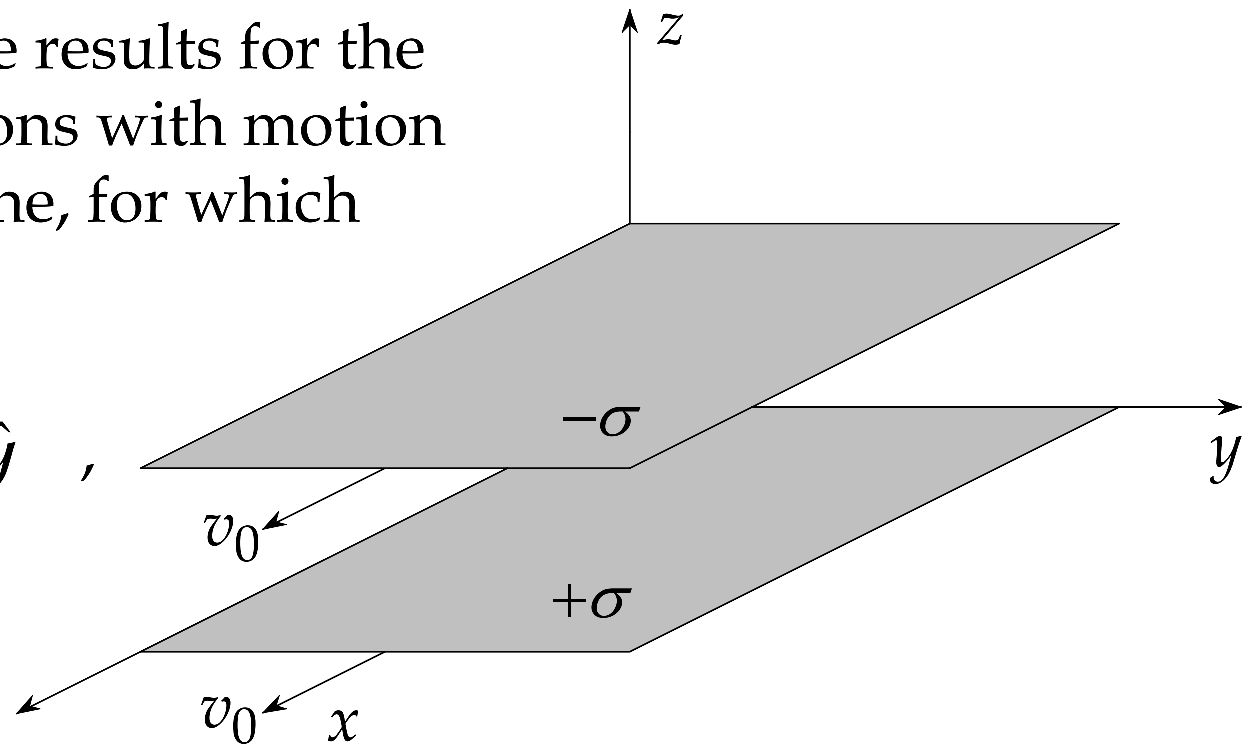
Planes infinite; only fragments shown.

Transformation of E and B in special relativity (continued)

We'll just sketch the results for the other two orientations with motion along x. First this one, for which

$$\boldsymbol{E} = 4\pi\sigma\hat{\boldsymbol{z}} = E_z\hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = -\frac{4\pi\sigma v_0}{c}\hat{\boldsymbol{y}} = B_y\hat{\boldsymbol{y}} \quad ,$$



in our frame. The charge and current density have the same magnitude in the moving observer's frame that they had before, so

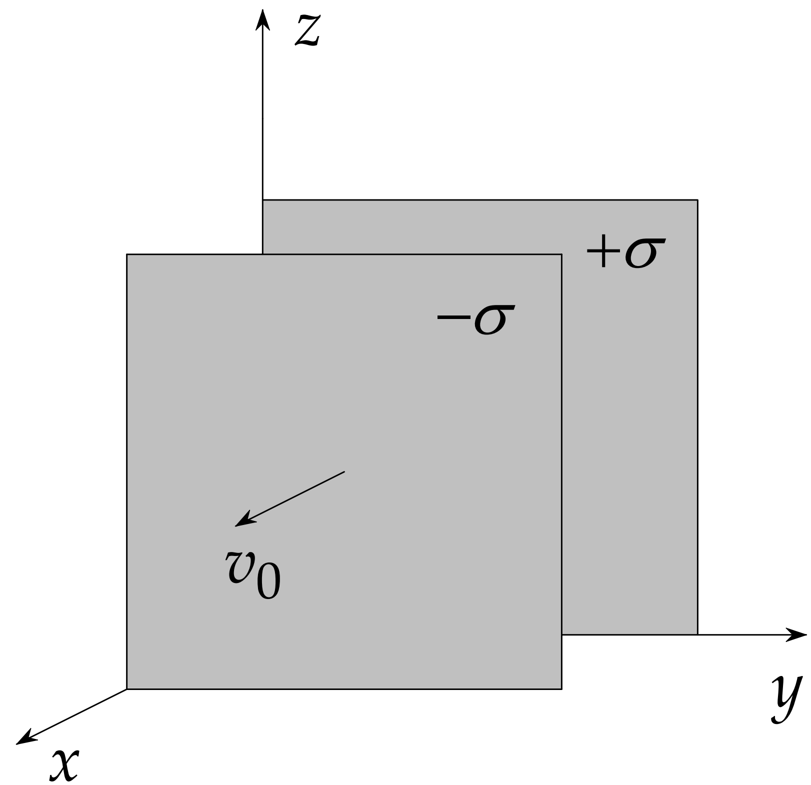
$$\overline{E}_z = \gamma\left(E_z + \beta B_y\right) \quad , \quad \overline{B}_y = \gamma\left(B_y + \beta E_z\right) \quad .$$

Transformation of *E* and *B* in special relativity (continued)

In this case we and the moving observer see the same charge and current density, so

$$\bar{E}_x = E_x \quad .$$

Unfortunately there isn't a current, or a magnetic field, in either frame. But we can produce a situation with constant magnetic field that will serve:



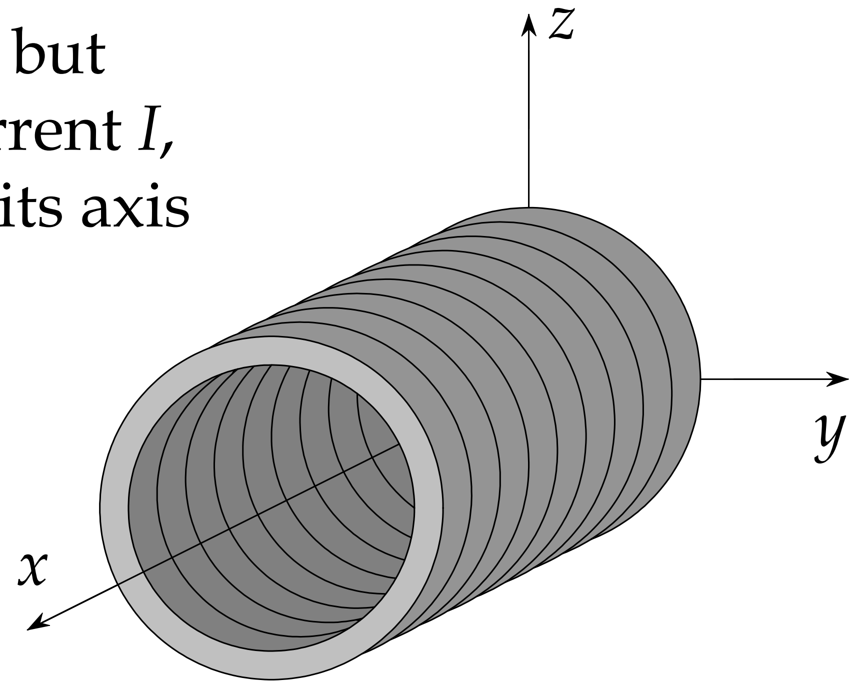
Transformation of E and B in special relativity (continued)

Same reference frames as before, but now a long solenoid carrying current I, with n turns per unit length and its axis along x, at rest in our reference frame:

$$\mathbf{B} = \frac{4\pi n I}{c}\hat{\mathbf{x}} = B_x \hat{\mathbf{x}} \quad .$$

The moving observer sees the length per turn contracted, and thus the turns per unit length increased:

$$\bar{n} = \gamma n \quad .$$



Solenoid infinite; only fragment shown.

Transformation of *E* and *B* in special relativity (continued)

But the moving observer also sees clocks in our frame running slowly (time dilation), so he/she sees a smaller current than we do:

$$\overline{I} = I/\gamma \quad ,$$

so

$$\overline{B}_x = \frac{4\pi\gamma n\left(I/\gamma\right)}{c} = \frac{4\pi n I}{c} = B_x \quad .$$

Collect them all:

$$\boxed{\begin{array}{lll} \overline{E}_x = E_x \quad , & \overline{E}_y = \gamma\left(E_y - \beta B_z\right) \quad , & \overline{E}_z = \gamma\left(E_z + \beta B_y\right) \quad , \\ \overline{B}_x = B_x \quad , & \overline{B}_y = \gamma\left(B_y + \beta E_z\right) \quad , & \overline{B}_z = \gamma\left(B_z + \beta E_y\right) \quad . \end{array}}$$

Transformation of *E* and *B* in special relativity (continued)

Or more compactly: if $\boldsymbol{v}$ is the velocity of the moving observer with respect to us, and $\boldsymbol{\beta} = \boldsymbol{v}/c$, then

$$\boxed{\begin{aligned} \overline{\boldsymbol{E}}_{\parallel} &= \boldsymbol{E}_{\parallel} \quad , \quad \overline{\boldsymbol{E}}_{\perp} = \gamma\left(\boldsymbol{E}_{\perp} + \boldsymbol{\beta} \times \boldsymbol{B}_{\perp}\right) \quad , \\ \overline{\boldsymbol{B}}_{\parallel} &= \boldsymbol{B}_{\parallel} \quad , \quad \overline{\boldsymbol{B}}_{\perp} = \gamma\left(\boldsymbol{B}_{\perp} - \boldsymbol{\beta} \times \boldsymbol{E}_{\perp}\right) \quad , \end{aligned}}$$

where $\parallel$ and $\perp$ mean parallel and perpendicular to $\boldsymbol{v}$.

- This transformation, in which the parallel components are left unchanged and the perpendicular components undergo a more complicated transformation, reminds one of the transformation of quantities like velocity and force, not like four-vectors – it isn't possible to construct a four-vector from ***E*** and/or ***B***.

Transformation of *E* and *B* in special relativity (continued)

Special cases:

- $\boldsymbol{B} = 0$ in our frame. Then, plugging into the "compact" result,

$$\overline{\boldsymbol{B}} = -\gamma\boldsymbol{\beta}\times\boldsymbol{E} \quad .$$

- $\boldsymbol{E} = 0$ in our frame. Then,

$$\overline{\boldsymbol{E}} = \gamma\boldsymbol{\beta}\times\boldsymbol{B} \quad .$$

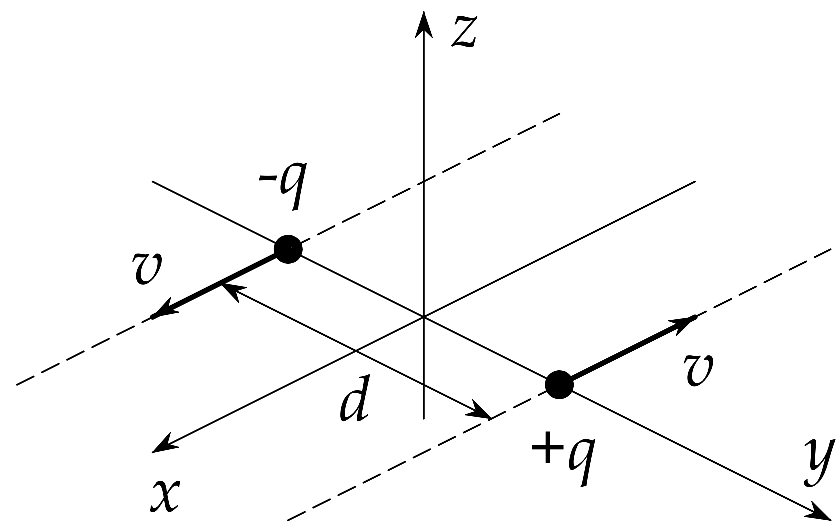
Example

Griffiths problem 12.45: Two charges, $\pm q$, are on parallel trajectories a distance d apart, moving with equal speeds v in opposite directions. We're interested in the force on $+q$ due to $-q$ at the instant they cross. Fill in the following table, doing all the consistency checks you can think of as you go along.

	Our rest frame (A)	$+q$ rest frame (B)	$-q$ rest frame (C)
E at $+q$ due to $-q$			
B at $+q$ due to $-q$			
F on $+q$ due to $-q$			

Example (continued)

Solution: In the rest frame of –q, the fields at and force on the other charge are easily obtained



$$\boldsymbol{E}_{\boldsymbol{C}} = -\frac{q}{d^2}\hat{\boldsymbol{y}} \quad ,$$

$$\boldsymbol{B}_{\boldsymbol{C}} = 0 \quad ,$$

$$\boldsymbol{F}_C = q\boldsymbol{E} = -\frac{q}{d^2}\hat{\boldsymbol{y}} \quad .$$

In our frame, for which $\gamma_A = \dfrac{1}{\sqrt{1 - v^2/c^2}}$,

$$\overline{E}_{y,A} = \gamma_A E_{y,C} = -\frac{q\gamma_A}{d^2} \quad , \quad \overline{B}_{z,A} = \gamma_A \beta E_{y,C} = -\frac{q\gamma_A\beta}{d^2} \quad ,$$

Example (continued)

and all other components zero; or

$$\overline{\boldsymbol{E}}_A = -\frac{q\gamma}{d^2}\hat{\boldsymbol{y}} \quad , \quad \overline{\boldsymbol{B}}_A = -\frac{q\gamma\beta}{d^2}\hat{\boldsymbol{z}} \quad .$$

The force in our frame is therefore

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{\beta}\times\boldsymbol{B} \quad ,$$

$$\boldsymbol{F_A} = -\frac{q^2\gamma_A}{d^2}\hat{\boldsymbol{y}} + \beta\frac{q^2\gamma_A\beta}{d^2}\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}} = -\frac{q^2\gamma_A}{d^2}\left(1+\beta^2\right)\hat{\boldsymbol{y}} \quad .$$

The velocity of +q relative to –q is

$$v_B = \frac{v+v}{1+v^2/c^2} = \frac{2v}{1+v^2/c^2} \quad ,$$

Example (continued)

so $$\gamma_B = \frac{1}{\sqrt{1-\left(\frac{2v/c}{1+v^2/c^2}\right)^2}} = \frac{1+v^2/c^2}{1-v^2/c^2} = \gamma_A^2\left(1+v^2/c^2\right) \quad .$$

Using the field transformation again, this time between frames C and B, we get

$$\overline{E}_{y,B} = \gamma_B E_{y,C} = -\frac{q\gamma_B}{d^2} \quad , \quad \overline{B}_z = \gamma_B \beta E_{y,C} = -\frac{q\gamma_B\beta}{d^2} \quad \text{, or}$$

$$\overline{\boldsymbol{E}}_B = -\frac{q}{d^2}\gamma_A^2\left(1+v^2/c^2\right)\hat{\boldsymbol{y}} \quad , \quad \overline{\boldsymbol{B}}_B = -\frac{q\beta}{d^2}\gamma_A^2\left(1+v^2/c^2\right)\hat{\boldsymbol{z}} \quad .$$

Example (continued)

In frame B the positive charge is not moving, so there's no magnetic force:

$$\boldsymbol{F}_B = q\overline{\boldsymbol{E}}_B = -\frac{q^2}{d^2}\gamma_A^2\left(1 + v^2/c^2\right)\hat{\boldsymbol{y}} \quad .$$

The electromagnetic field tensor

So we can't make a four-vector out of ***E*** or ***B***. What can we make?

- ❑ The fields have six independent components that transform in a way rather more complicated than four-vectors do.
- ❑ What kinds of tensors have this many independent components?
  - Second-rank four-tensors have sixteen.
  - Symmetric ones have ten.
  - **Antisymmetric** ones have six:

$$t_{\mu\nu} = -t_{\nu\mu} \quad .$$

…so the diagonal elements are zero.

The electromagnetic field tensor (continued)

- Second-rank four-tensors Lorentz-transform with two applications of the matrix:

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\kappa} t^{\kappa\lambda} \Lambda^{\nu}_{\lambda} \quad ,$$

where, for motion along x as we considered when deriving the field transformations,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiply it out: for example,

The electromagnetic field tensor (continued)

$$
\begin{aligned}
\overline{t}^{11} &= \Lambda_0^1\Lambda_0^1 t^{00} + \Lambda_0^1\Lambda_1^1 t^{01} + \Lambda_1^1\Lambda_0^1 t^{10} + \Lambda_1^1\Lambda_1^1 t^{11} \\
&= \Lambda_0^1\Lambda_1^1 t^{01} + \Lambda_1^1\Lambda_0^1 t^{10} \\
&= -\gamma^2\beta\left(t^{01} + t^{10}\right) = 0 \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\overline{t}^{10} &= \Lambda_0^1\Lambda_0^0 t^{00} + \Lambda_0^1\Lambda_1^0 t^{01} + \Lambda_1^1\Lambda_0^0 t^{10} + \Lambda_1^1\Lambda_1^0 t^{11} \\
&= \Lambda_0^1\Lambda_1^0 t^{01} + \Lambda_1^1\Lambda_0^0 t^{10} \\
&= (\gamma\beta)^2\, t^{01} + \gamma^2 t^{10} = \gamma^2\left(1-\beta^2\right)t^{10} = t^{10} \quad .
\end{aligned}
$$

- And so on. Here's the full set:

The electromagnetic field tensor (continued)

$$\overline{t}^{01} = t^{01}, \quad \overline{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right), \quad \overline{t}^{03} = \gamma\left(t^{03} + \beta t^{13}\right)$$

$$\overline{t}^{23} = t^{23}, \quad \overline{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right), \quad \overline{t}^{32} = \gamma\left(t^{32} - \beta t^{02}\right) \quad .$$

Miraculously, that's just like the way the fields transform, if we identify the components of an **electromagnetic field tensor**, $F^{\mu\nu}$, as follows:

$$F^{01} = E_x, \quad F^{02} = E_y, \quad F^{03} = E_z,$$

$$F^{12} = B_z, \quad F^{13} = -B_y, \quad F^{23} = B_x.$$

The electromagnetic field tensor (continued)

In full matrix regalia,

$$F = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad .$$