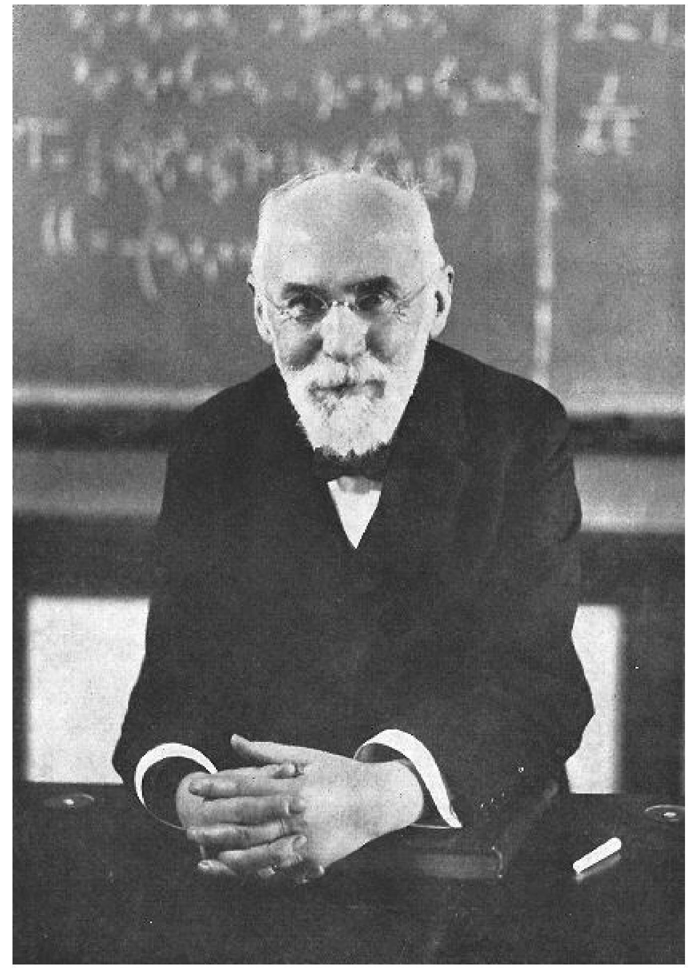
Today in Physics 218: relativistic electrodynamics in tensor form

- The dual field tensor
- Charge and current densities, the Maxwell equations, and the Lorentz force, in tensor form
- The four-potential and gauge transformations
- The relativistic analogue of the inhomogeneous wave equation for potentials.



Henrik Lorentz, his blackboard covered with what looks like E&M equations in tensor form.

The dual electromagnetic field tensor

Last time we constructed a second-rank four-tensor for the electric and magnetic fields:

$$F^{\mu\nu} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad ,$$

which we got by comparing the Lorentz transformation of the elements of an antisymmetric second-rank four-tensor with that of the electric and magnetic fields:

The dual electromagnetic field tensor (continued)

$$\bar{E}_x = E_x \quad , \quad \bar{E}_y = \gamma\left(E_y - \beta B_z\right) \quad , \quad \bar{E}_z = \gamma\left(E_z + \beta B_y\right) \quad ,$$

$$\bar{B}_x = B_x \quad , \quad \bar{B}_y = \gamma\left(B_y + \beta E_z\right) \quad , \quad \bar{B}_z = \gamma\left(B_z - \beta E_y\right) \quad ;$$

$$\bar{t}^{01} = t^{01}, \quad \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right), \quad \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{13}\right) \quad ,$$

$$\bar{t}^{23} = t^{23}, \quad \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right), \quad \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right) \quad .$$

We picked

$$F^{01} = E_x, F^{02} = E_y, F^{03} = E_z, F^{12} = B_z, F^{13} = -B_y, F^{23} = B_x.$$

But we could just as well have picked

$$F^{01} = B_x, F^{02} = B_y, F^{03} = B_z, F^{12} = -E_z, F^{13} = E_y, F^{23} = -E_x.$$

The dual electromagnetic field tensor (continued)

This makes a different-looking tensor that is called the **dual** of F:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} .$$

that, yet, embodies the same physics as F. Sometimes it's more convenient to use one than the other, so it's handy to have both around, as we'll see in a minute.

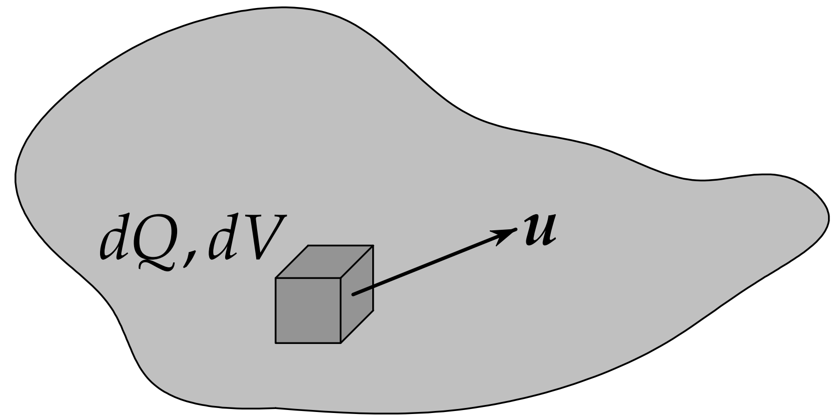
The dual electromagnetic field tensor (continued)

Both F and G come in contravariant and covariant forms. Elements with covariant indices differ in sign for the zeroth component, compared to the contravariant form:

$$F_{\mu\nu} = \begin{pmatrix} -F^{00} & -F^{01} & -F^{02} & -F^{03} \\ -F^{10} & F^{11} & F^{12} & F^{13} \\ -F^{20} & F^{21} & F^{22} & F^{23} \\ -F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} .$$

The four-current-density

Consider a cloud of electric charge, flying by at velocity $\boldsymbol{u}$. In our frame the charge and current density represented by an infinitesimal element within the cloud is



$$\rho = \frac{dQ}{dV} \quad , \quad \boldsymbol{J} = \rho \boldsymbol{u} \quad .$$

In the element's rest frame, though,

$$\rho_0 = \frac{dQ}{dV_0} = \sqrt{1 - u^2/c^2}\,\frac{dQ}{dV} \quad ,$$

because of the Lorentz contraction of one dimension of the infinitesimal volume as seen from our frame.

The four-current-density (continued)

Thus, in terms of the "proper" charge density $\rho_0 = dQ/dV_0$,

$$\rho = \frac{\rho_0}{\sqrt{1-u^2/c^2}} = \rho_0 \frac{\eta^0}{c} \quad , \quad \boldsymbol{J} = \frac{\rho_0 \boldsymbol{u}}{\sqrt{1-u^2/c^2}} = \rho_0 \boldsymbol{\eta} \quad ,$$

where we have noted the presence of the four-velocity η^μ. Since proper charge density is invariant (there's only one rest frame for a given object), the charge and current density above comprise a four-vector, just as the four-velocity does:

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \quad .$$

The four-current-density (continued)

Charge and current density are related (in a given reference frame) by

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

$$= \sum_{i=0}^{3} \frac{\partial J^i}{\partial x^i} + \frac{1}{c}\frac{\partial J^0}{\partial t} = \sum_{i=0}^{3} \frac{\partial J^i}{\partial x^i} + \frac{\partial J^0}{\partial x^0} \quad , \text{ or}$$

$$\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0} \quad .$$

Relativistic continuity equation

Aside: the four-gradient

The four-gradient with respect to contravariant coordinates, $\partial/\partial x^{\mu}$, turns out to be a covariant four-vector operator, as might be worth showing here. To show that it is covariant, it suffices to show that, for any scalar function *f*, $\partial f/\partial x^{\mu}$ transforms like a covariant four vector. This just requires the chain rule and the Lorentz transformation for relative motion along the *x* direction:

$$\frac{\partial f}{\partial \bar{x}^0} = \frac{1}{c}\frac{\partial f}{\partial \bar{t}} = \frac{1}{c}\left(\frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{t}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{t}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{t}}\right) \quad .$$

But $t = \gamma\left(\bar{t} + v\bar{x}/c^2\right)$ and $x = \gamma\left(\bar{x} + v\bar{t}\right)$, so

$$\frac{\partial t}{\partial \bar{t}} = \gamma \quad , \quad \frac{\partial x}{\partial \bar{t}} = \gamma v \quad , \quad \frac{\partial y}{\partial \bar{t}} = \frac{\partial z}{\partial \bar{t}} = 0 \quad .$$

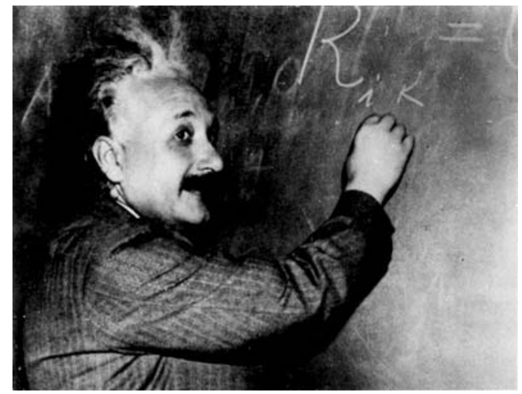
Aside: the four-gradient (continued)

Thus
$$\frac{\partial f}{\partial \bar{x}^0} = \frac{1}{c}\gamma\left(\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}\right) = \gamma\left(\frac{\partial f}{\partial x^0} + \beta\frac{\partial f}{\partial x}\right) \quad .$$

Similarly,
$$\frac{\partial f}{\partial \bar{x}^1} = \frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{x}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{x}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{x}}$$
$$= \frac{\partial f}{\partial t}\frac{\gamma v}{c^2} + \frac{\partial f}{\partial x}\gamma = \gamma\left(\frac{\partial f}{\partial x^1} + \beta\frac{\partial f}{\partial x^0}\right) \quad ,$$
$$\frac{\partial f}{\partial \bar{x}^2} = \frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{y}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{y}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x^2} ,$$
$$\frac{\partial f}{\partial \bar{x}^3} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x^3} \quad .$$

Aside: the four-gradient (continued)

But this is how a covariant vector is supposed to transform.



Contravariant:	Covariant:
$\overline{a}^0 = \gamma\left(a^0 - \beta a^1\right)$	$-\overline{a}_0 = \gamma\left(-a_0 - \beta a_1\right)$
$\overline{a}^1 = \gamma\left(a^1 - \beta a^0\right)$	$\overline{a}_1 = \gamma\left(a_1 + \beta a_0\right)$
$\overline{a}^2 = a^2$	$\overline{a}_2 = a_2$
$\overline{a}^3 = a^3$	$\overline{a}_3 = a_3$

Remember to change signs for your covariant zeroth components!

So $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$ is covariant. Similarly (Griffiths problem 12.55), the other form, $\partial^\mu \equiv \dfrac{\partial}{\partial x_\mu}$, is contravariant.

The Maxwell equations in tensor form

Note that

$$\partial_\nu F^{0\nu} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla \cdot \boldsymbol{E} \quad \left(= 4\pi\rho = \frac{4\pi}{c} J^0 \right) \quad ,$$

$$\partial_\nu F^{1\nu} = -\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left[-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{B} \right]_x \quad \left(= \frac{4\pi}{c} J_x \right).$$

Similarly,

$$\partial_\nu F^{2\nu} = \left[-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{B} \right]_y \quad \left(= \frac{4\pi}{c} J_y \right) \quad ,$$

$$\partial_\nu F^{3\nu} = \left[-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{B} \right]_z \quad \left(= \frac{4\pi}{c} J_z \right) \quad .$$

The Maxwell equations in tensor form (continued)

In other words, both Gauss's and Ampère's laws are contained in the expression

$$\boxed{\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \quad .}$$

Similarly,

$$\partial_\nu G^{0\nu} = \nabla \cdot \boldsymbol{B} \quad (=0) \quad ,$$

$$\partial_\nu G^{i\nu} = -\left[\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}\right]_i \quad (=0) \quad ,$$

so Faraday's law and the "no monopoles" law are

$$\boxed{\partial_\nu G^{\mu\nu} = 0 \quad .}$$

One can see how this notation might save some writing.

The Lorentz force in tensor form

This turns out simply to be

$$K^{\mu} = \frac{q\eta_{\nu}}{c} F^{\mu\nu} \quad .$$

To wit:

$$K^{1} = \frac{q\eta_{\nu}}{c} F^{1\nu} = \frac{q}{c}\left(-\eta^{0}F^{10} + \eta^{1}F^{11} + \eta^{2}F^{12} + \eta^{3}F^{13}\right)$$

$$= \frac{q}{c}\left[\left(-c\gamma_{u}\right)\left(-E_{x}\right) + u_{y}\gamma_{u}B_{z} + u_{z}\gamma_{u}\left(-B_{y}\right)\right] = q\gamma_{u}\left(\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right)_{x} \quad .$$

Similarly,

$$K^{2} = q\gamma_{u}\left(\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right)_{y} \quad , \quad K^{3} = q\gamma_{u}\left(\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right)_{z} \quad .$$

The Lorentz force in tensor form (continued)

Thus
$$\boldsymbol{K} = q\gamma_u\left(\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right) \quad ,$$

or, since $\boldsymbol{K} = \gamma_u\boldsymbol{F}$, as we saw in lecture on 16 April,
$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right) \quad ,$$

as usual.

- The zeroth component of K turns out to give (Griffiths problem 12.54):
$$K^0 = \frac{q}{c^2}\gamma_u\boldsymbol{u}\cdot\boldsymbol{E} \quad .$$

Since $K^0 = \dfrac{dp^0}{d\tau} = \dfrac{d}{d\tau}\dfrac{W}{c} = \dfrac{\gamma_u}{c}\dfrac{dW}{dt}$, (lecture, 16 April), this just means that power is force times velocity.

The four-potentials and gauge transformations

Unlike the fields, and as you might expect – given the relationship of V and $\mathbf{A}$ to the energy and momentum per unit charge – the potentials can be combined into a four-vector:

$$A^{\mu} = (V, \mathbf{A}) \quad .$$

It turns out that, just as the fields are easily expressed as vector derivatives of the potentials in "nonrelativistic" E&M, the field tensor is easily expressed in terms of derivatives of the four-potential.

- Consider simply the antisymmetric combination:

$$\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

The four-potentials and gauge transformations (continued)

- The 01 element of this is

$$\partial^0 A^1 - \partial^1 A^0 = \frac{\partial A_x}{\partial x_0} - \frac{\partial A_0}{\partial x_1} = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial V}{\partial x}$$

$$= \left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V \right)_x = E_x \quad .$$

  Similarly, the 02 and 03 elements give the other two components of ***E***.

- The 12 element is

$$\partial^1 A^2 - \partial^2 A^1 = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$= \left(\nabla \times \boldsymbol{A} \right)_z = B_z \quad .$$

The four-potentials and gauge transformations (continued)

- Similarly, the 13 and 23 elements give the other two components of ***B***. Evidently,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad .$$

Note that the Lorentz gauge condition – the one that's the most use in wave equations for the potentials – can also be written easily in these terms:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial V}{\partial (ct)} = \partial_\nu A^\nu \quad .$$

The relativistic analogue of the inhomogeneous wave equation for potentials

Earlier this semester, we found that elimination of the fields in favor of the potentials turned some of the Maxwell equations into inhomogeneous wave equations in the components of the potential. That works here too:

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu$$

$$\partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu =$$

In the first term, switch the order of differentiation and apply the Lorentz gauge condition:

$$\partial^\mu \left(\cancel{\partial_\nu A^\nu}\right) - \partial_\nu \partial^\nu A^\mu = -\partial_\nu \partial^\nu A^\mu \quad \text{, or}$$

$$\boxed{\Box^2 A^\mu \equiv \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} J^\mu} \quad .$$

We’re done.

That’s it for new material. The final two class meetings will be reviews.