Physics 218 Practice Midterm Exam: Solutions

Spring 2004

If any of these solutions seems obscure, please contact us so we can explain it better.

Problem 1 (20 points)

In a certain material, the group velocity of electromagnetic waves is twice the phase velocity. Derive an expression for these waves's wavenumber, k, as a function of their angular frequency, ω.

$$v_g = 2v \quad \Rightarrow \quad \frac{d\omega}{dk} = 2\frac{\omega}{k}$$

$$\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega'} = 2\int_{k_0}^{k} \frac{dk'}{k'}$$

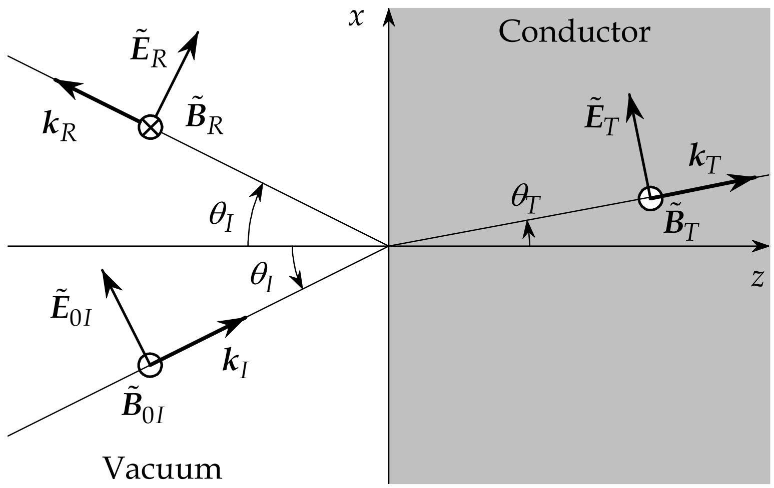
$$\ln\frac{\omega}{\omega_0} = 2\ln\frac{k}{k_0}$$

$$\frac{\omega}{\omega_0} = \left(\frac{k}{k_0}\right)^2 \quad \Rightarrow \quad \boxed{k = k_0\sqrt{\frac{\omega}{\omega_0}}}\ .$$

Here ω_0 and k_0 are constants, the evaluation of which would require more information about the material. (One could have done with just one integration constant, of course, but the formula looks more symmetrical this way.)

Problem 2 (40 points)

Light, propagating in vacuum, is incident obliquely (incidence angle θ_I) on the planar surface of a good conductor $\left(\sigma \gg \varepsilon\omega/4\pi\right.$, or $\sigma \gg \varepsilon_r\varepsilon_0\omega$ in MKS units$\left.\right)$ *for which* $\varepsilon = \mu = 1\left(\varepsilon = \varepsilon_0, \mu = \mu_0\right.$ in MKS$\left.\right)$. *The light is polarized, with the electric field in the plane of incidence.*



a. *What is the angle of the transmitted wavevector with respect to the normal? (Hint: it's not equal to θ_I.)*

In a good conductor,

$$\tilde{k}_T = \frac{\sqrt{2\pi\omega\sigma}}{c}(1+i) \quad \Rightarrow \quad v_T = \frac{\omega}{\tilde{k}_T} = \frac{c}{\tilde{n}_T} \quad \text{, so}$$

$$\tilde{n}_T = \frac{c\tilde{k}_T}{\omega} = \sqrt{\frac{2\pi\sigma}{\omega}}(1+i) \quad \text{, and}$$

$$\sin\theta_T = \frac{\sin\theta_I}{\tilde{n}_T} = \frac{\sin\theta_I}{1+i}\sqrt{\frac{\omega}{2\pi\sigma}} \to 0 \quad ,$$

since by assumption the conductivity is large. (Never mind the fact that's it's complex; it's still small enough to consider zero to be a good approximation.) Thus,

$$\boxed{\theta_T = 0} \quad .$$

b. *Using the electromagnetic boundary conditions, write down enough equations to determine the reflected and transmitted electric field amplitudes. (Don't solve them yet.)*

Recall that $\boldsymbol{B} = \hat{\boldsymbol{k}} \times \frac{c\tilde{k}}{\omega}\boldsymbol{E}$ in a conductor, and that $\boldsymbol{H} = \boldsymbol{B}/\mu = \boldsymbol{B}$ here:

$\boldsymbol{E}_{\parallel}$ is continuous: $\quad \tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_I = \tilde{E}_{0T}\cos\theta_T = \tilde{E}_{0T} \quad .$

$\boldsymbol{H}_{\parallel}$ is continuous: $\quad \tilde{E}_{0I} - \tilde{E}_{0R} = \frac{c\tilde{k}_T}{\omega}\tilde{E}_{0T} \quad .$

Or, with $\alpha = 1/\cos\theta_I$ and $\beta = c\tilde{k}_T/\omega = \sqrt{2\pi\sigma/\omega}\,(1+i)$,

$$\boxed{\begin{aligned}\tilde{E}_{0I} + \tilde{E}_{0R} &= \alpha\tilde{E}_{0T} \\ \tilde{E}_{0I} - \tilde{E}_{0R} &= \beta\tilde{E}_{0T}\end{aligned}}\ ,\ .$$

The answer is the same in MKS, if you take $\beta = \sqrt{\sigma/2\omega}\,(1+i)$ instead.

c. *Solve for the reflected electric field amplitude, in terms of the incident amplitude, and from this derive the time-averaged intensity reflection coefficient* $\rho = I_R/I_I$.

Multiply the first boundary condition through by β/α and subtract:

$$\left(\frac{\beta}{\alpha} - 1\right)\tilde{E}_{0I} + \left(\frac{\beta}{\alpha} + 1\right)\tilde{E}_{0R} = 0 \quad \Rightarrow \quad \boxed{\tilde{E}_{0R} = \tilde{E}_{0I}\frac{\alpha - \beta}{\alpha + \beta}}\ .$$

The intensity reflection coefficient is therefore

$$\rho = \frac{\tilde{E}_{0R}\tilde{E}_{0R}^*}{\tilde{E}_{0I}\tilde{E}_{0I}^*} = \boxed{\frac{\alpha - \beta}{\alpha + \beta}\frac{\alpha^* - \beta^*}{\alpha^* + \beta^*}}\ ,$$

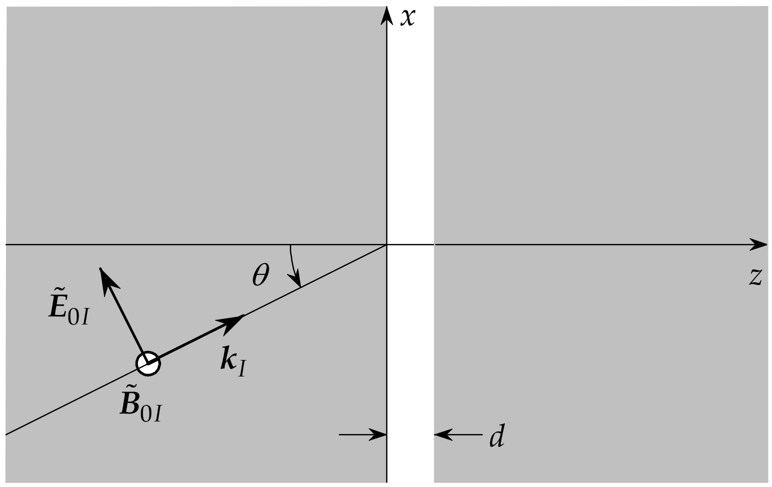
again with $\alpha = 1/\cos\theta_I$ and $\beta = c\tilde{k}_T/\omega = \sqrt{2\pi\sigma/\omega}\,(1+i)$.

d. *Is there a "Brewster's angle" for reflection from conductors? Explain why or why not.*

In dielectrics, $\rho = 0$ at $\theta_I = \theta_B$. Here $\rho = 0$ only if $(\alpha - \beta)(\alpha^* - \beta^*) = 0$, and this in turn only gives a single value of α (and θ_I) if $\beta = \beta^*$; that is, if β is real. But β *can't* be real for a conductor, so $\rho \neq 0$ at all incidence angles: there's no Brewster angle in this sense. (There does turn out to be a minimum in the intensity reflection coefficient, though.)

Problem 3 (40 points)

Frustrated total reflection. *Light with angular frequency ω, propagating in glass (n = 1.5) and polarized in the plane of incidence, encounters a plane-parallel, vacuum-filled gap at an angle* θ *greater than* $\arcsin(1/n)$ *-- that is, an angle at which total reflection would be expected for a single surface. Glass fills the whole Universe, apart from the gap.*



Calculate the amplitude of the electric field transmitted through the gap. Is it possible to make the gap transparent, at this frequency?

Call the left and right glass regions 0 and 2 respectively, and the gap 1. The admittances of these regions, for TM waves as shown in the figure, are

$$Y_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{\cos\theta_I} = \frac{n}{\cos\theta_I} = Y_2 \quad ,$$
$$Y_1 = \frac{1}{\cos\theta_T} \quad ,$$

(using cgs units, mind you; the 0 subscripts just indicate the medium containing the incident light) where $\cos\theta_T$ is determined from Snell's Law:

$$\sin\theta_T = n\sin\theta_I$$
$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = \sqrt{1-n^2\sin^2\theta_I} = i\sqrt{n^2\sin^2\theta_I - 1} \equiv i\alpha \quad ,$$

and is purely imaginary. The phase delay for light propagating across the vacuum gap is

$$\delta_1 = kd\cos\theta_T = \frac{\omega d\cos\theta_T}{c} = i\frac{\omega d}{c}\sqrt{n^2\sin^2\theta_I - 1} \equiv i\phi \quad ,$$

and is also purely imaginary. Thus

$$\cos\delta_1 = \frac{1}{2}\left[e^{-\phi} + e^{\phi}\right] = \cosh\phi \quad ,$$
$$\sin\delta_1 = \frac{1}{2i}\left[e^{-\phi} - e^{\phi}\right] = -i\sinh\phi \quad .$$

This is all we need to know for the elements of the characteristic matrix of the vacuum gap:

$$M_1 = \begin{bmatrix} \cos\delta_1 & -i\cos\theta_T \sin\delta_1 \\ -i\dfrac{\sin\delta_1}{\cos\theta_T} & \cos\delta_1 \end{bmatrix} = \begin{bmatrix} \cosh\phi & -i\alpha\sinh\phi \\ \dfrac{i\sinh\phi}{\alpha} & \cosh\phi \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad .$$

Thus the amplitude transmission coefficient t is

$$\begin{aligned} t &= \frac{2Y_0}{m_{11}Y_0 + m_{12}Y_0Y_2 + m_{21} + m_{22}Y_2} \\ &= \frac{2\dfrac{n}{\cos\theta_I}}{\cosh\phi\dfrac{n}{\cos\theta_I} - i\alpha\sinh\phi\left(\dfrac{n}{\cos\theta_I}\right)^2 + \dfrac{i\sinh\phi}{\alpha} + \cosh\phi\dfrac{n}{\cos\theta_I}} \\ &= \frac{2}{2\cosh\phi - i\left(\dfrac{n\alpha}{\cos\theta_I} - \dfrac{\cos\theta_I}{n\alpha}\right)\sinh\phi} \quad . \end{aligned}$$

Everything besides the i in this expression is real, so it's not too hard to get the intensity transmission coefficient:

$$\begin{aligned} \tau = \frac{Y_2}{Y_0}tt^* &= \frac{2}{2\cosh\phi - i\left(\dfrac{n\alpha}{\cos\theta_I} - \dfrac{\cos\theta_I}{n\alpha}\right)\sinh\phi}\,\frac{2}{2\cosh\phi + i\left(\dfrac{n\alpha}{\cos\theta_I} - \dfrac{\cos\theta_I}{n\alpha}\right)\sinh\phi} \\ &\boxed{= \frac{1}{\cosh^2\phi + \dfrac{1}{4}\left(\dfrac{n\alpha}{\cos\theta_I} - \dfrac{\cos\theta_I}{n\alpha}\right)\sinh^2\phi}} \;, \end{aligned}$$

with ϕ and α as given above. (Same in cgs and MKS.)

The only way in general for the gap to be transparent – i.e. for $\tau = 1$ – is for $\phi = \frac{\omega d}{c}\sqrt{n^2\sin^2\theta_I - 1} = 0$, because $\cosh(0) = 1$ and $\sinh(0) = 0$. This in turn can be true for finite ω if d = 0 (no gap), so it *isn't* possible to make the gap transparent. (Interestingly, $\tau = 1$ for any finite d if $\sin\theta_I = 1/n$. This pathological case, however, requires that the glass-vacuum planes be infinite, which is why the problem was specified with $\sin\theta_I > 1/n$.)

Don't worry. This solution is indeed a little too long for the problem to be put on a real exam.

Problem 4 (20 points)

Consider using, as a gauge condition,

$$V = 0 \quad \text{everywhere,}$$
$$\boldsymbol{A} \neq 0 \quad .$$

Show explicitly that this can describe electromagnetic waves in vacuum; that is, find the appropriate ***A****, and show that it gives electric and magnetic fields with the correct properties of waves in a vacuum.*

There are more elegant solutions to this problem than the following, but this is simplest in concept. If $V = 0$, then $\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$. Let's take $\boldsymbol{E} = \boldsymbol{E}_0 \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)$, which is obviously a solution to the wave equation for $\boldsymbol{E}$; or, with the right choice of axes, $\boldsymbol{E} = \boldsymbol{E}_0 \cos\left(kz - \omega t\right)$, with $\boldsymbol{E}_0$ perpendicular to the z axis. Then,

$$\boldsymbol{A} = -c\boldsymbol{E}_0 \int \cos\left(kz - \omega t\right) dt = \boxed{\frac{c}{\omega}\boldsymbol{E}_0 \sin\left(kz - \omega t\right)} + \text{constant} \quad .$$

We can take the constant to be zero and still get the same fields, as we're about to see: the magnetic field would thus be

$$\begin{aligned}
\boldsymbol{B} &= \nabla \times \boldsymbol{A} = \nabla \times \frac{c}{\omega}\boldsymbol{E}_0 \sin\left(kz - \omega t\right) \\
&= \frac{c}{\omega}\left(-kE_{0y}\cos\left(kz - \omega t\right)\hat{\boldsymbol{x}} + kE_{0x}\cos\left(kz - \omega t\right)\hat{\boldsymbol{y}}\right) \\
&= \left(-E_{0y}\hat{\boldsymbol{x}} + E_{0x}\hat{\boldsymbol{y}}\right)\cos\left(kz - \omega t\right) = \hat{\boldsymbol{z}} \times \boldsymbol{E}_0 \cos\left(kz - \omega t\right) \quad .
\end{aligned}$$

This, of course, is well known to be the plane-wave solution to the wave equation for $\boldsymbol{B}$. So the gauge choice correctly gives the properties of plane waves. But we showed earlier this semester than any electromagnetic wave in vacuum is a superposition of plane waves (Griffiths problem 9.4, on problem set 2). Thus this gauge is good for all electromagnetic waves in vacuum.