Physics 218 Practice Final Exam: Solutions

Spring 2004

Problems 2 (a and b), 3, 4, and 5 (a and c) were on the real final exam last time I taught PHY 218. I already used the remaining problems on that exam for the practice midterm.

Problem 1 (50 points)

a. *A spherical shell (inside radius a, outside radius b) has charge $Q\cos\omega t$ spread on its outer surface, and charge $-Q\cos\omega t$ spread uniformly on its inner surface. What is the electric field in the far-field zone, a distance $r \gg \lambda \gg b$ from the center of the sphere?*

   Both spheres act as if their charge is concentrated at their centers. The total charge is zero; so are all of the multipole moments, because the positive and negative charges are located in exactly the same place. Because of the latter, there is *no* radiation; $E = 0$ everywhere in the far field.

b. *Another shell is the same in all respect to the first one, except that the centers of the inner and outer surfaces are displaced with respect to one another by a distance d. What is the electric field in the far-field zone, a distance $r \gg \lambda \gg b$ from the center of the sphere?*

   Now there's a finite dipole moment:

$$p = \hat{z}Qd\cos\omega t \quad ,$$

   where $\hat{z}$ points along the line between the centers, from minus to plus. The system thus radiates like a dipole, and in the far field domain the electric field is given as usual by

$$E = -\hat{\theta}\frac{Qd\omega^2 \sin\theta}{rc^2}\cos\omega\left(t - r/c\right)$$

   in cgs units. (Swap the $1/c^2$ for $\mu_0/4\pi$ to get the answer in MKS units.)

Problem 2 (50 points)

a. *Calculate the distance y_0 from the center of a rain drop, at which the "rainbow ray" is incident: light that corresponds exactly to the scattering-angle extremum that in turn corresponds to the primary rainbow. Refer to the diagram at right for the geometry.*
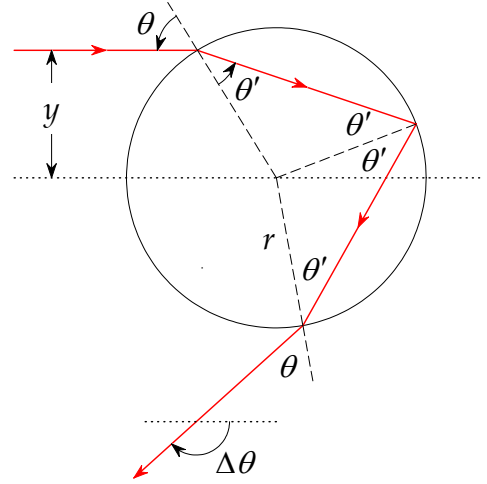
In the coordinate system of the diagram, in which z is the initial direction of the light,

$$\frac{dz}{dy} = \frac{y}{\sqrt{r^2 - y^2}} = \tan\theta \quad,$$

so

$$\sin\theta = \frac{y}{\sqrt{\left(r^2 - y^2\right) + y^2}} = \frac{y}{r} \quad,$$

$$\cos\theta = -\frac{\sqrt{r^2 - y^2}}{\sqrt{\left(r^2 - y^2\right) + y^2}} = -\sqrt{1 - \frac{y^2}{r^2}} \quad.$$

Snell's Law implies that

$$\sin\theta' = \frac{1}{n}\sin\theta = \frac{y}{nr} \quad.$$

Thus the scattering angle is, as a function of y,

$$\Delta\theta = 2\theta - 4\theta' + \pi = 2\arcsin\left(\frac{y}{r}\right) - 4\arcsin\left(\frac{y}{nr}\right) + \pi \quad.$$

Find the minimum as usual:

$$\frac{d}{dy}\Delta\theta = 2\frac{d}{dy}\arcsin\left(\frac{y}{r}\right) - 4\frac{d}{dy}\arcsin\left(\frac{y}{nr}\right)$$

$$= \frac{2}{\sqrt{r^2 - y^2}} - \frac{4}{\sqrt{n^2 r^2 - y^2}} = 0 \quad\text{at } y = y_1;$$

$$n^2 r^2 - y_1^2 = 4\left(r^2 - y_1^2\right)$$

$$y_1 = \frac{r}{3}\sqrt{12 - 3n^2} \quad.$$

b.   *Repeat the calculation of part a, but for the secondary rainbow. Again, refer to the diagram at right for the geometry.*

For the secondary rainbow,

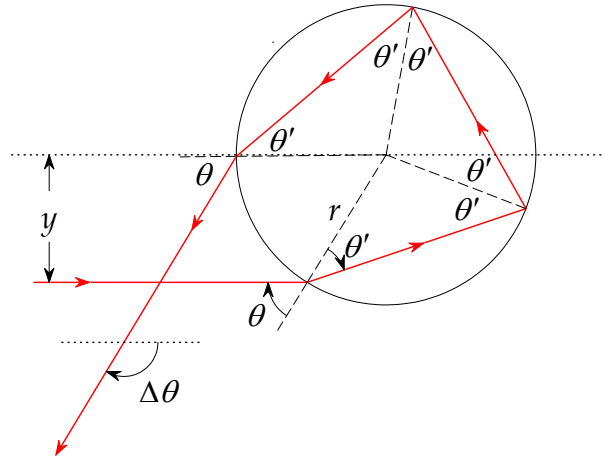$$\Delta\theta = 2\arcsin\left(\frac{y}{r}\right) - 6\arcsin\left(\frac{y}{nr}\right) \quad,$$

and so

$$\frac{d}{dy}\Delta\theta = \frac{2}{\sqrt{r^2 - y^2}} - \frac{6}{\sqrt{n^2 r^2 - y^2}} = 0$$

at $y = y_2$;

$$n^2 r^2 - y_2^2 = 9\left(r^2 - y_2^2\right)$$

$$y_2 = -r\sqrt{\frac{9 - n^2}{8}} \quad.$$

c. *In the Bible (Genesis 9:13-15), it is written that shortly after the Flood subsided, God said to Noah and his family,*

> *I have set my bow in the clouds, and it shall be a sign of the covenant between me and the earth. Henceforth when I bring clouds over the Earth and the bow is seen in the clouds, I will remember my covenant that is between me and you and every living creature of all flesh; and the waters shall never again become a flood to destroy all flesh.*

So before the Flood, raindrops did not produce rainbows, but afterward they did.

Describe how the refractive index of water would have had to change during this conversation, in order for the optical properties of raindrops to change like this.

I think this is the earliest literary reference to rainbows; had to work it in to the course somehow…

Arguing from the results of parts a and b (which would certainly be good enough to get full credit): There would be no rainbows, of at least the primary and secondary sort, if n were large enough that y_1 and y_2 came out imaginary, which is the case for $n > 3$. So the index had to start off greater than 3, and drop suddenly to its present value near 4/3.

If you want to be really clean about it, note that the Nth rainbow lies at

$$\Delta\theta = 2\theta - 2(N+1)\theta' + N\pi \qquad (N = 1, 2, \ldots)$$

$$= 2\arcsin\left(\frac{y}{r}\right) - 2(N+1)\arcsin\left(\frac{y}{nr}\right) + N\pi \quad,$$

such that

$$\frac{d}{dy}\Delta\theta = \frac{2}{\sqrt{r^2 - y^2}} - \frac{2(N+1)}{\sqrt{n^2 r^2 - y^2}} = 0 \quad \text{at } y = y_N;$$

$$n^2 r^2 - y_N^2 = (N+1)^2 \left(r^2 - y_N^2\right)$$

$$y_N^2 \left[(N+1)^2 - 1\right] = r^2 \left[(N+1)^2 - n^2\right]$$

$$y_N = \pm r \sqrt{\frac{(N+1)^2 - n^2}{(N+1)^2 - 1}} \quad .$$

Thus there is no N^{th} rainbow if $n \geq N+1$. Since in principle one could consider N as high as one likes, n would formally have to be infinite in order not to have rainbows of *any* order. In practice, though, it would suffice to have n simply a good deal larger than 1 (say, 10), in order that the high-order rainbows still allowed would be very faint, owing to the large reflectivity of the drop at high index.

Problem 3 (50 points)

Diffraction of a Gaussian beam. *The electric field in a laser beam, as the beam leaves the end of the laser, is linearly polarized vertically, is axially symmetric, and has a magnitude which depends upon distance from the laser's axis as follows:*

$$E = E_0 e^{-s^2/s_0^2} = E_0 e^{-\left(x'^2 + y'^2\right)/s_0^2} \quad .$$

The laser beam is pointed perpendicular to a screen which lies a very long distance r $\left(\gg s_0\right)$ away from the laser. What is the electric field on this screen, as a function of distance $q = \sqrt{x^2 + y^2}$ from the point on the screen at which the laser is aimed?
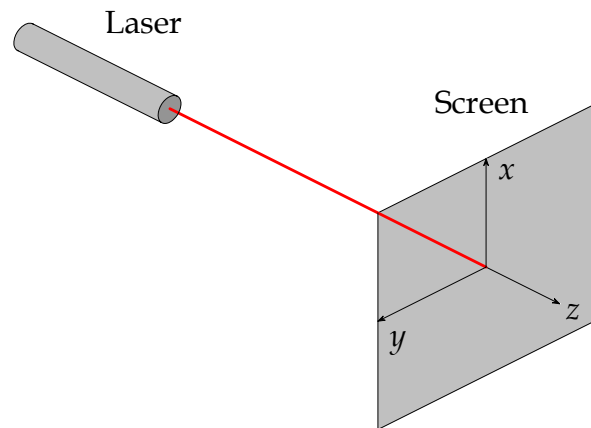
Make a rough plot of the electric field amplitude as a function of q.

Hint: work in Cartesian coordinates initially, and complete the square in the exponent of the integrand, to carry out the integral.

$$E_N = E_0 e^{-\left(x'^2+y'^2\right)/s_0^2} e^{-i\omega t} \quad \text{(a Gaussian)},$$

$$E_F = \frac{e^{i(kr-\omega t)}}{\lambda r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_0 e^{-\left(x'^2+y'^2\right)/s_0^2} e^{-i\left(k_x x'+k_y y'\right)} dx' dy' \quad .$$

Note that

$$-\frac{x'^2}{s_0^2} - ik_x x' = -\left(\frac{x'^2}{s_0^2} + ik_x x' + \left[\frac{ik_x s_0}{2}\right]^2\right) + \left[\frac{ik_x s_0}{2}\right]^2$$

$$= -\left(\frac{x'}{s_0} + \frac{ik_x s_0}{2}\right)^2 - \frac{k_x^2 s_0^2}{4} \quad .$$

Similarly for y'. The integral becomes

$$E_F = \frac{E_0 e^{i(kr-\omega t)}}{\lambda r} e^{-\left(k_x^2+k_y^2\right)s_0^2/4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(\frac{x'}{s_0} + \frac{ik_x s_0}{2}\right)^2} e^{-\left(\frac{y'}{s_0} + \frac{ik_y s_0}{2}\right)^2} dx' dy' \quad .$$

Now change variables:

$$u = \frac{x'}{s_0} + \frac{ik_x s_0}{2} \quad du = \frac{dx'}{s_0} \quad -\infty < u < \infty$$

$$v = \frac{y'}{s_0} + \frac{ik_y s_0}{2} \quad dv = \frac{dy'}{s_0} \quad -\infty < v < \infty$$

$$E_F = \frac{s_0^2 E_0 e^{i(kr-\omega t)}}{\lambda r} e^{-\left(k_x^2+k_y^2\right)s_0^2/4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(u^2+v^2\right)} du\, dv \quad ,$$

and change them again:

$$\left.\begin{array}{l} \rho^2 = u^2 + v^2 \\ \tan\phi = v/u \end{array}\right\} \quad du\, dv = \rho\, d\rho\, d\phi \quad , \quad 0 < \rho < \infty, 0 < \phi < 2\pi;$$

$$E_F = \frac{s_0^2 E_0 e^{i(kr-\omega t)}}{\lambda r} e^{-\left(k_x^2+k_y^2\right)s_0^2/4} \int_0^{2\pi} d\phi \int_0^{\infty} \rho e^{-\rho^2} d\rho \quad ,$$

and change one of them yet again, and the integrals become trivial:

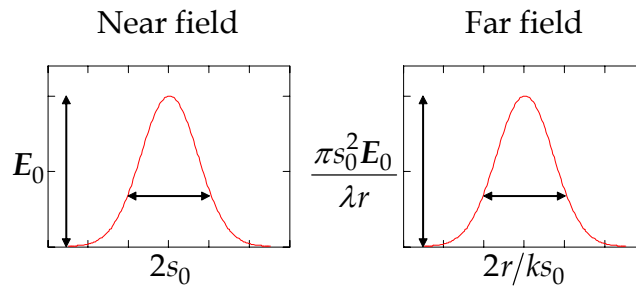$$w = \rho^2 \quad , \quad dw = 2\rho\, d\rho \quad , \quad 0 < w < \infty \quad ,$$

$$E_F = \frac{s_0^2 E_0 e^{i(kr-\omega t)}}{2\lambda r} e^{-\left(k_x^2+k_y^2\right)s_0^2/4} \int_0^{2\pi} d\phi \int_0^{\infty} e^{-w} dw$$

$$= \frac{\pi s_0^2 E_0 e^{i(kr-\omega t)}}{\lambda r} e^{-\left(k_x^2+k_y^2\right)s_0^2/4} \quad .$$

We are asked to note that $k_x^2 + k_y^2 = k^2\left(x^2/r^2 + y^2/r^2\right) = k^2 q^2/r^2$, which makes it

$$E_F = \frac{\pi s_0^2 E_0 e^{i(kr-\omega t)}}{\lambda r} e^{-k^2 q^2 s_0^2/4r^2} \quad .$$

This is also a Gaussian. Gaussian beams stay Gaussian as they propagate, because the Fourier transform of a Gaussian is another Gaussian.

Near field   Far field

Problem 4 (50 points)

Electric quadrupole radiation. *Two oscillating electric dipoles, separated by a distance d, are oriented as shown in the figure at right. Using what you know about the potentials for individual dipoles, calculate the scalar potential V in the far field* $\left(r \gg \lambda \gg d\right)$.

Hints: *Keep only terms that are first order in d. Note that neither dipole lies at the origin.*

Superpose the retarded potentials from the two dipoles, which are

$$V_\pm = \mp\frac{p_0 \omega}{c}\frac{\cos\theta_\pm}{\imath_\pm}\sin\omega\left(t - \frac{\imath_\pm}{c}\right) \quad .$$
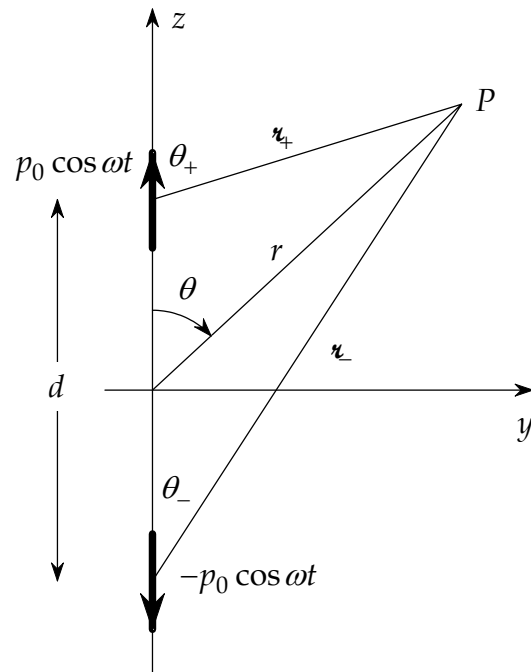
We can use the law of cosines and the binomial theorem to obtain approximations for $\imath_\pm$:

$$\imath_\pm = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 \mp 2r\frac{d}{2}\cos\theta} \cong r\sqrt{1 \mp \frac{d}{r}\cos\theta} \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right) \quad,$$

$$\frac{1}{\imath_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad,$$

to first order in d. Note also from the diagram – specifically, the triangle sides that lie along the z axis – that

$$\imath_\pm \cos\theta_\pm \pm \frac{d}{2} = r\cos\theta \quad,$$

or

$$\cos\theta_\pm = \frac{r\cos\theta \mp d/2}{\imath_\pm} \cong r\left(\cos\theta \mp \frac{d}{2r}\right)\frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\cong \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r} \quad \left(\text{ignoring terms in } d^2!\right)$$

$$= \cos\theta \mp \frac{d}{2r}\left(1 - \cos^2\theta\right) = \cos\theta \mp \frac{d}{2r}\sin^2\theta \quad,$$

also to first order in d. Thus the retarded-time factor becomes

$$\sin\omega\left(t - \frac{\imath_\pm}{c}\right) = \sin\omega\left[t - \frac{r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right] = \sin\left(\omega t_0 \pm \frac{\omega d}{2c}\cos\theta\right)$$

$$= \sin\omega t_0 \cos\left(\frac{\omega d}{2c}\cos\theta\right) \pm \cos\omega t_0 \sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

$$\cong \sin\omega t_0 \pm \frac{\omega d}{2c}\cos\theta\cos\omega t_0 \quad,$$

where we have abbreviated $t_0 = t - r/c$, and have used the small-angle approximation for the terms in $\omega d/2c$, to first order in d.

Putting all this together, we get, for the retarded potentials of the two dipoles,

$$V_\pm = \mp\frac{p_0\omega}{c}\frac{\cos\theta_\pm}{\imath_\pm}\sin\omega\left(t - \frac{\imath_\pm}{c}\right)$$

$$= \mp\frac{p_0\omega}{c}\left(\cos\theta \mp \frac{d}{2r}\sin^2\theta\right)\frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)\left(\sin\omega t_0 \pm \frac{\omega d}{2c}\cos\theta\cos\omega t_0\right) \quad.$$

When multiplying it out, remember to drop terms of order d^2 or higher:

$$V_\pm = \mp \frac{p_0 \omega}{cr}\left(\cos\theta \mp \frac{d}{2r}\sin^2\theta \pm \frac{d}{2r}\cos^2\theta \right)\left(\sin\omega t_0 \pm \frac{\omega d}{2c}\cos\theta\cos\omega t_0 \right)$$

$$= \mp \frac{p_0 \omega}{cr}\left(\cos\theta\sin\omega t_0 \pm \frac{\omega d}{2c}\cos^2\theta\cos\omega t_0 \pm \frac{d}{2r}\left(\cos^2\theta - \sin^2\theta\right)\sin\omega t_0 \right) \ .$$

Now we can finally do the superposition:

$$V = V_+ + V_-$$

$$= -\frac{p_0 \omega}{cr}\left(\cos\theta\sin\omega t_0 + \frac{\omega d}{2c}\cos^2\theta\cos\omega t_0 + \frac{d}{2r}\left(\cos^2\theta - \sin^2\theta\right)\sin\omega t_0 \right)$$

$$+ \frac{p_0 \omega}{cr}\left(\cos\theta\sin\omega t_0 - \frac{\omega d}{2c}\cos^2\theta\cos\omega t_0 - \frac{d}{2r}\left(\cos^2\theta - \sin^2\theta\right)\sin\omega t_0 \right)$$

$$= -\frac{p_0 \omega}{cr}\left(\frac{\omega d}{c}\cos^2\theta\cos\omega t_0 + \frac{d}{r}\left(\cos^2\theta - \sin^2\theta\right)\sin\omega t_0 \right)$$

$$= -\frac{p_0 \omega^2 d}{c^2 r}\left(\cos^2\theta\cos\omega t_0 + \frac{c}{\omega r}\left(\cos^2\theta - \sin^2\theta\right)\sin\omega t_0 \right) \ .$$

Since in the far field we have $r \gg c/\omega$, we can neglect the second term in the brackets compared to the first one, and obtain a rather simple expression:

$$V = -\frac{p_0 \omega^2 d}{c^2}\frac{\cos^2\theta}{r}\cos\omega t_0 = -\frac{p_0 \omega^2 d}{c^2}\frac{\cos^2\theta}{r}\cos\omega\left(t - \frac{r}{c} \right) \ .$$

Multiply the right side by $1/4\pi\varepsilon_0$ to get the MKS version of the answer.

Problem 5 (50 points)

a. *Using the electromagnetic field tensor $F^{\mu\nu}$ and the dual tensor $G^{\mu\nu}$, show that*

$$E^2 - B^2 \quad \text{(cgs units) or} \quad E^2 - c^2 B^2 \quad \text{(MKS units)}$$

and

$$\mathbf{E}\cdot\mathbf{B}$$

are invariant under Lorentz transformations.

There are three invariants one can make by the inner products of $F^{\mu\nu}$ and $G^{\mu\nu}$. Note while doing the sums that a zero covariant index brings in an extra minus sign, and that the diagonal elements are zero:

$$F_{\mu\nu}F^{\mu\nu} = -F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} - F^{10}F^{10} + F^{12}F^{12} + F^{13}F^{13}$$
$$- F^{20}F^{20} + F^{21}F^{21} + F^{23}F^{23} - F^{30}F^{30} + F^{31}F^{31} + F^{32}F^{32}$$
$$= -E_x^2 - E_y^2 - E_z^2 - E_x^2 + B_z^2 + B_y^2 - E_y^2 + B_z^2 + B_x^2 - E_x^2 + B_y^2 + B_x^2$$
$$= 2\left(E^2 - B^2\right) \quad ;$$

$$G_{\mu\nu}G^{\mu\nu} = -G^{01}G^{01} - G^{02}G^{02} - G^{03}G^{03} - G^{10}G^{10} + G^{12}G^{12} + G^{13}G^{13}$$
$$- G^{20}G^{20} + G^{21}G^{21} + G^{23}G^{23} - G^{30}G^{30} + G^{31}G^{31} + G^{32}G^{32}$$
$$= -B_x^2 - B_y^2 - B_z^2 - B_x^2 + E_z^2 + E_y^2 - B_y^2 + E_z^2 + E_x^2 - B_z^2 + E_y^2 + E_x^2$$
$$= 2\left(B^2 - E^2\right) \quad ;$$

$$F_{\mu\nu}G^{\mu\nu} = -F^{01}G^{01} - F^{02}G^{02} - F^{03}G^{03} - F^{10}G^{10} + F^{12}G^{12} + F^{13}G^{13}$$
$$- F^{20}G^{20} + F^{21}G^{21} + F^{23}G^{23} - F^{30}G^{30} + F^{31}G^{31} + F^{32}G^{32}$$
$$= -E_x B_x - E_y B_y - E_z B_z - E_x B_x - B_z E_z - B_y E_y$$
$$- E_y B_y - B_z E_z - B_x E_x - E_z B_z - B_y E_y - B_x E_x$$
$$= -4E \cdot B \quad .$$

The first two results each show that $E^2 - B^2$ is Lorentz invariant. The third shows that $E \cdot B$ is invariant. (Q.E.D.) It works similarly in MKS units except for the stray factors of c.

b. *Suppose that, in a certain inertial frame S, the electric field **E** and the magnetic field **B** are neither parallel nor perpendicular. Show that in a different inertial frame $\bar{S}$, moving relative to S at velocity **v** given by*

$$v = \frac{c}{\gamma^2}\frac{E \times B}{B^2 + E^2} \quad \text{(cgs units)} \quad \text{or} \quad \frac{1}{\gamma^2}\frac{E \times B}{B^2 + E^2/c^2} \quad \text{(MKS units),} \quad ,$$

the fields $\bar{E}$ and $\bar{B}$ are parallel.

Suppose the two frames move relative to one another along the x direction, so we can use our usual forms of the Lorentz transformation. If x is also the direction of $E \times B$, as we are asked to show, then E and B lie in the y-z plane. Let ψ be the angle between the two fields in S, and for simplicity let one of them, say E, point along z; then $B_z = B\cos\psi$ and $B_y = B\sin\psi$. Thus the relativistic transformations of the fields give us

$$\overline{E}_y = \gamma(-\beta B \cos\psi) \quad , \quad \overline{E}_z = \gamma(E + \beta B \sin\psi) \quad ,$$
$$\overline{B}_y = \gamma(B \sin\psi + \beta E) \quad , \quad \overline{B}_z = \gamma(B \cos\psi) \quad .$$

The x components of the field are still zero; this component is left unchanged by the transformation. $\overline{E}$ and $\overline{B}$ are parallel if the ratios of their y and z components are equal:

$$\frac{\overline{E}_y}{\overline{E}_z} = -\frac{\gamma\beta B \cos\psi}{\gamma(E + \beta B \sin\psi)} = \frac{\overline{B}_y}{\overline{B}_z} = \frac{\gamma(B \sin\psi + \beta E)}{\gamma B \cos\psi} \quad .$$

Multiply it out:

$$-\beta B^2 \cos^2\psi = (E + \beta B \sin\psi)(B \sin\psi + \beta E)$$
$$= \beta E^2 + \beta B^2 \sin^2\psi + (1 + \beta^2) EB \sin\psi \quad ;$$

$$\beta E^2 + \beta B^2 (\cos^2\psi + \sin^2\psi) + (1 + \beta^2) EB \sin\psi = 0$$

$$\beta = -\frac{(1 + \beta^2) EB \sin\psi}{E^2 + B^2} = -\frac{1}{\gamma^2} \frac{|\mathbf{E} \times \mathbf{B}|}{E^2 + B^2} \quad .$$

The way we've set up the coordinate axes, though, the cross product of the fields is

$$\mathbf{E} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & E \\ 0 & B \sin\psi & B \cos\psi \end{vmatrix} = \hat{x} \begin{vmatrix} 0 & E \\ B \sin\psi & B \cos\psi \end{vmatrix} = -\hat{x} EB \sin\psi$$

(I guess that means I should have pointed $\mathbf{B}$ along the z axis), so the minus signs cancel:

$$v = -\frac{c}{\gamma^2} \frac{\mathbf{E} \times \mathbf{B}}{E^2 + B^2} \quad , \text{ q.e.d.}$$

c.   *Is there a frame in which the electric and magnetic fields are perpendicular?*

No. Since the fields are not perpendicular in the original frame, $\mathbf{E} \cdot \mathbf{B} \neq 0$, and since this quantity is Lorentz-invariant, they can't be perpendicular in any other frame either.

Problem 6 (50 points)

Unpolarized light with angular frequency ω is incident from vacuum, at angle θ, on a planar conducting surface. Calculate the degree of polarization,

$$\delta = \frac{I_\perp - I_\parallel}{I_\perp + I_\parallel} \quad ,$$

of reflected light, and describe in a few words the state of the light's polarization: nature, magnitude and direction. (Here $\perp$ and $\parallel$ mean perpendicular and parallel to the plane of incidence.)

We need to handle the TE ($\perp$) and TM ($\parallel$) parts of the incident light separately. TE first: the boundary conditions on the components of **E** and **H** parallel to the surface are

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\frac{1}{\mu_1}\left(-\sqrt{\mu_1 \varepsilon_1}\tilde{E}_{0I}\cos\theta_I + \sqrt{\mu_1 \varepsilon_1}\tilde{E}_{0R}\cos\theta_I\right) = -\frac{1}{\mu_2}\frac{c\tilde{k}_2}{\omega}\tilde{E}_{0T}\cos\theta_T \quad ,$$
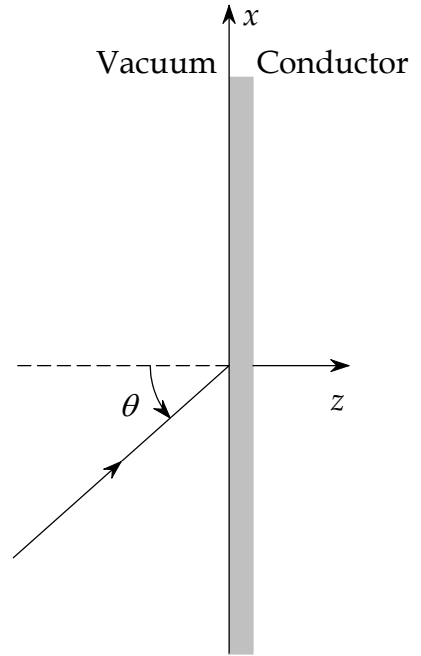
which we can rearrange in a familiar way:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad ,$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \left(\frac{\cos\theta_T}{\cos\theta_I}\right)\left(\sqrt{\frac{\mu_1}{\varepsilon_1}}\frac{c\tilde{k}_2}{\mu_2\omega}\right)\tilde{E}_{0T} = \alpha\tilde{\beta}\tilde{E}_{0T} \quad .$$

Multiply the first condition by $\alpha\tilde{\beta}$ and subtract:

$$\left(\alpha\tilde{\beta} - 1\right)\tilde{E}_{0I} + \left(\alpha\tilde{\beta} + 1\right)\tilde{E}_{0R} = 0 \quad ;$$

$$\left(\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right)_\perp = \frac{1 - \alpha\tilde{\beta}}{1 + \alpha\tilde{\beta}} \quad .$$

And now for TM, for which the boundary conditions on the components of **E** and **H** parallel to the surface are

$$\tilde{E}_{0I} \cos\theta_I + \tilde{E}_{0R} \cos\theta_I = \tilde{E}_{0T} \cos\theta_T \quad,$$

$$\frac{1}{\mu_1}\left(\sqrt{\mu_1\varepsilon_1}\tilde{E}_{0I} - \sqrt{\mu_1\varepsilon_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\frac{c\tilde{k}_2}{\omega}\tilde{E}_{0T} \quad,$$

which can also be rearranged in a familiar way, using the same α and $\tilde{\beta}$, and solved just as before:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \frac{\cos\theta_T}{\cos\theta_I}\tilde{E}_{0T} = \alpha\tilde{E}_{0T} \quad,$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\frac{c\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0T} = \tilde{\beta}\tilde{E}_{0T} \quad,$$

$$\left(\frac{\tilde{\beta}}{\alpha}-1\right)\tilde{E}_{0I} + \left(\frac{\tilde{\beta}}{\alpha}+1\right)\tilde{E}_{0R} = 0 \quad;$$

$$\left(\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right)_{\parallel} = \frac{\alpha-\tilde{\beta}}{\alpha+\tilde{\beta}} \quad.$$

Now,

$$\left(\frac{I_R}{I_I}\right)_{\perp} \pm \left(\frac{I_R}{I_I}\right)_{\parallel} = \frac{1-\alpha\tilde{\beta}}{1+\alpha\tilde{\beta}}\frac{1-\alpha\tilde{\beta}^*}{1+\alpha\tilde{\beta}^*} \pm \frac{\alpha-\tilde{\beta}}{\alpha+\tilde{\beta}}\frac{\alpha-\tilde{\beta}^*}{\alpha+\tilde{\beta}^*}$$

$$= \frac{1-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\alpha^2\beta^2}{1+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\alpha^2\beta^2} \pm \frac{\alpha^2-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\beta^2}{\alpha^2+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\beta^2} \quad;$$

$$\delta = \frac{I_{\perp}-I_{\parallel}}{I_{\perp}+I_{\parallel}} = \frac{\dfrac{1-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\alpha^2\beta^2}{1+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\alpha^2\beta^2} - \dfrac{\alpha^2-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\beta^2}{\alpha^2+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\beta^2}}{\dfrac{1-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\alpha^2\beta^2}{1+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\alpha^2\beta^2} + \dfrac{\alpha^2-\alpha\tilde{\beta}-\alpha\tilde{\beta}^*+\beta^2}{\alpha^2+\alpha\tilde{\beta}+\alpha\tilde{\beta}^*+\beta^2}} \quad.$$

This is an ugly expression, but at normal incidence $(\alpha = 1)$ it simplifies considerably:

$$\delta = \frac{\dfrac{1-\tilde{\beta}-\tilde{\beta}^*+\beta^2}{1+\tilde{\beta}+\tilde{\beta}^*+\beta^2} - \dfrac{1-\tilde{\beta}-\tilde{\beta}^*+\beta^2}{1+\tilde{\beta}+\tilde{\beta}^*+\beta^2}}{\dfrac{1-\tilde{\beta}-\tilde{\beta}^*+\beta^2}{1+\tilde{\beta}+\tilde{\beta}^*+\beta^2} + \dfrac{1-\tilde{\beta}-\tilde{\beta}^*+\beta^2}{1+\tilde{\beta}+\tilde{\beta}^*+\beta^2}} = 0 \quad;$$

unpolarized light reflected at normal incidence is still unpolarized. At oblique incidence it obviously doesn't vanish, so the reflected light is polarized.

I wouldn't expect you to go any farther than this, on a real test. (I wish I had written the problem just in terms of the fields; I didn't think the result was going to be this ugly.) It turns out that the polarization is slight (a few percent at most), and perpendicular to the plane of incidence; that is, δ turns out to be small and positive.