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Control of entanglement and the high-entanglement limit

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We examine two-particle entanglement and ways in which it might be coherently controlled. Control is desirable for successful implementation of quantum computing. As one result, we find that reaching very high entanglement may be feasible experimentally.

Keywords: high entanglement; Schmidt number; Schmidt mode; continuous entanglement; entanglement control

1. Introduction

It is well known by now that the non-locality associated with entangled quantum states is the key to performing computational tasks that cannot be realized classically (Nielsen & Chuang 2000; Williams & Clearwater 1998, 2000). We have been studying two-particle entanglement in a variety of physical contexts, in order to gain a better physical understanding of ways in which entanglement might be controlled coherently. This will be a desirable element in the successful implementation of quantum computing.

Four examples of two-particle entanglement are shown in figure 1. The first is Franck–Hertz scattering, which was initially observed experimentally at a time before the quantum theory of matter had even been formalized. An incoming electron scatters from an electron bound in an atom, say hydrogen for simplicity, and an electron is observed coming out. There is no way to tell which electron comes out, and the quantum state describing the two is an entangled state. The time evolution of this two-particle system has been described using a simplified atom model in Grobe et al. (1994). Entangled particles do not have to be massive, of course, and one of the two-particle quantum systems that has been studied most intensely recently is the signal–idler pair of photons emitted in spontaneous parametric down-conversion. These two photons may be entangled in wavelength, momentum, angular momentum and frequency, as well as polarization. The process of down-conversion entanglement has been examined in various contexts: Einstein–Podolsky–Rosen (EPR; Einstein et al. 1935) analogues in quadrature amplitudes (Reid 1989; Reid & Drummond 1988),

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quantum information (Huang & Eberly 1993; Law et al. 2000), angular momentum (Arnaut & Barbosa 2000), and quantum pattern formation (Lugiato et al. 2001).

Another example in figure 1 shows the ejection of correlated electrons from an initially entangled two-electron ground state, driven by a very intense short-pulse laser field, an area of great experimental interest due to the anomalously high yield of double ionization. Theoretical analyses are beginning to unravel this mystery (Liu et al. 1999). The final example will be described in more detail below. It involves recoil-induced entanglement between a massive atom and the photon it has just spontaneously emitted (Chan et al. 2002).

2. Schmidt modes

The Schmidt decomposition theorem is the simplest analytical tool available for detailed study of entangled two-particle dynamics. Its utility is limited because it can be applied only to two-particle pure states. However, it allows one to characterize any bipartite pure state as a unique pairwise association of eigenstates (Schmidt modes) of the subsystem density matrices. Thus its results have clear physical interpretations in terms of projective measurements. This is valuable enough to justify the attention paid to it. The Schmidt decomposition is a matter of relatively simple matrix manipulations if the state space is small enough, but a sketch of the theorem indicates its greater generality. For an introduction, see Ekert & Knight (1995) and Eberly et al. (2003).

The desired result of a Schmidt decomposition is achieved when a two-particle state, $|\Psi_{AB}\rangle$, is expressed as a single sum over basis states (the Schmidt modes) of the two subsystems:

$$|\Psi_{AB}\rangle = \sum_n \sqrt{\lambda_n} |\phi_n^A\rangle \otimes |\phi_n^B\rangle. \quad (2.1)$$

We call these basis states the ‘information eigenstates’ for the problem, reflecting their definition as eigenstates of the density matrices for subsystems ‘A’ and ‘B’. The $\lambda$ are the corresponding eigenvalues, which are the same for the two subsystems.
The eigenvalues are naturally all positive and sum to unity. They can be ordered according to \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \), and the degree of entanglement of the two systems is obviously related to the number of \( \lambda \) that contribute significantly to the Schmidt summation.

As a numerical measure of a degree of entanglement (Grobe et al. 1994) we will use the so-called Schmidt number, or participation ratio, denoted by \( K \):

\[
K \equiv \frac{1}{\sum_n \lambda_n^2} \geq 1.
\] (2.2)

Naturally, if one of the \( \lambda \) is unity, then the others must vanish and \( K = 1 \), so the Schmidt sum has only one term, and the state is not entangled. On the other hand, for example, if there are \( N \) states all with \( \lambda_n = 1/N \), then \( K = N \). Consequently, this gives \( K \) a loose physical interpretation as the number of Schmidt states that are significant in making up the entangled two-particle state \(|\Psi_{AB}\rangle\). Note also that the number of eigenvalues is given by the dimension of the subsystem’s Hilbert space that is the larger, but there are only as many non-zero eigenvalues as the smaller dimension. Thus, entanglement of two quantum systems, one of them with a two-dimensional state space, can produce \( K \) values no greater than \( K = 2 \), no matter how large the dimension of the other state space. It is clear that a Bell state

\[
|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}}(|(+)_A, (-)_B\rangle \pm |(-)_A, (+)_B\rangle)
\]

has \( K = 2 \), and one can refer to higher values of \( K \) as indicating beyond-Bell entanglement.

3. Quantum entanglement in very large state spaces

When the state space for two quantum systems increases, the information-carrying capacity increases as well. Systems described by continuous variables have infinite state spaces. We have used the Schmidt decomposition (Chan et al. 2002; Grobe et al. 1994; Huang & Eberly 1993; Law et al. 2000; Liu et al. 1999) to study all of the examples shown in figure 1, and have found values of \( K = 4-5 \), only a few times higher than the Bell-state value. In principle, much higher values are available, and as one type of entanglement-control question, we ask how they can be reached.

We will begin here with the entanglement that arises between an atom and a spontaneously emitted photon, which we have treated previously (Chan et al. 2002). An idealized experimental set-up, requiring detection of photon and atom, is sketched in figure 2, similar to the experiments reported by Kurtsiefer et al. (1997).

For simplicity we will fix the positions of the atom and photon detectors back to back, because then we expect the atom to suffer the strongest recoil. Experimental set-ups with similar implications for entanglement were employed by Pfau et al. (1994) and Chapman et al. (1995). They go beyond spontaneous emission in the sense that they involve Rayleigh or Raman processes, but the entanglement due to momentum conservation is of the same type.

The Hamiltonian for any of these radiative interactions of a two-level atom is conventional. The expression for the quantum state following emission has been given by Rzążewski & Żakowicz (1992) as

\[
|\Psi(t)\rangle = \int d^3q \int d^3k C(q,k)e^{-i(E_q/h+k)c t}|q,g\rangle \otimes |1_k\rangle,
\] (3.1)
where we denote the state of the decayed atom with final momentum $h\mathbf{q}$ by $|q,g\rangle$ and that of the emitted photon with wavevector $\mathbf{k}$ by $|1\rangle$. We assume that the atom centre of mass was initially prepared thermally, so the wave-vector (velocity) distribution for the centre of mass of the atom adds a Gaussian factor with width $\sigma$.

Then the joint amplitude $C(q,k)$ gives us the correlation between the atom and the photon in the form

$$C(q,k) = Ne^{-\frac{(q-k)^2}{\sigma^2}}\frac{1}{(h/2M)(2qk-k^2) + kc - \omega_0 + i\gamma},$$

(3.2)

which in this case is not separable, i.e. not factorable in the form $C(q,k) = F(q)G(k)$. The constant $N$ is a normalization factor which incorporates the coupling, and is a slowly varying function of $k$, as compared with the Lorentzian denominator.

We now follow Chan et al. (2002) and introduce a pair of scaled variables, motivated by the strong inequalities $Mc^2 \gg h\omega_0$, $h\sigma c \gg h\gamma$. The amplitude can then be transformed into

$$C(q,k) = Ne^{-\frac{(Mc\delta q/\omega_0)^2}{2}}\frac{1}{\delta k + \delta q + i\gamma},$$

(3.3)

where the scaled wave vectors are

$$\delta q \equiv \frac{h\omega_0}{Mc^2}(qc - \omega_0) \quad \text{and} \quad \delta k \equiv \left( kc - \omega_0 + \frac{hk_0^2}{2M} \right).$$

(3.4)

Our goal is to rewrite the amplitude in Schmidt form,

$$C(q,k) = \sum_n \sqrt{\lambda_n} u_n(k)v_n(q),$$

(3.5)

which is typically possible only numerically, but here one obtains (Chan et al. 2002) an unexpected analytic bonus from the Gaussian exponential function. The combination of parameters in the exponent suggests an ‘observability’ index, $\eta$, where

$$\eta \equiv \frac{\omega_0 h\sigma}{\gamma Mc} = \frac{\Delta \omega}{\gamma},$$

(3.6)
in which $\Delta \omega = \omega_0 (\hbar \sigma/Mc) = (\Delta v/c)\omega_0$ is interpreted as the thermal (motional) line broadening. Remarkably, the observability index $\eta$ has a clear interpretation as the ratio of motional linewidth to natural linewidth. It is found numerically that $K \approx 1 + 0.28(\eta - 1)$ for $\eta > 1$, as shown in figure 3.

The amplitude for atomic spontaneous emission is an example of ‘cross-modular’ entanglement of two different quantum entities, where one entangled partner is easily detectable and the other is a high-speed information carrier. This kind of entanglement is interesting, as it may be used in building photonics devices that are useful in quantum information processing. However, here we are focused on control of entanglement, and in particular in increasing it. The realistic values for $K$ that can be reached via momentum entanglement in pure spontaneous emission are very small, of the order of 1, but in the case of Rayleigh and Raman scattering the values may be very large.

To see this, we note that we have shown in (3.6) that the control parameter $\eta$ is proportional to the radiative lifetime $\tau = 2/\gamma$. By modifying the lifetime, $\eta$ could be controlled, and $K$ with it. Raman scattering controls the effective lifetime by controlling the population of the decaying level. First note that the atomic upper level is taken to be initially 100% occupied in spontaneous emission, but in Raman scattering the steady-state occupation probability is reduced by the factor $(\Omega/2\Delta)^2$, where $\Omega$ and $\Delta$ are the Rabi frequency and the detuning of the incident field. In this way, in effect, the radiative lifetime will be increased by the factor $(2\Delta/\Omega)^2$, which in turn gives an enhancement for $\eta$ by the same factor, which could well be several orders of magnitude. For a specific illustration of an experiment to achieve very high entanglement, we propose Raman scattering involving the hyperfine ground levels $F = 1$ and $F = 2$ and a D-line transition in the sodium atom, as sketched in figure 4.
Figure 4. A sketch of the Raman transition proposed for a demonstration of high entanglement.

In analysing the Raman case it is important to recognize that the approximations made in our calculations imply the simultaneous validity of two oppositely directed constraints:

(i) the atom should remain in the excitation region long enough to reach steady state, but

(ii) the effective Raman linewidth (half the inverse lifetime) should be very sharp on the scale of recoil energies.

These conditions potentially conflict with each other if pushed too far, but appear to remain compatible in an experimentally accessible regime associated with a sodium atom cooled to $T \approx 0.01\, \text{K}$. This fixes $\sigma$ via the sodium mass and pins the atom RMS speed at $v \approx 2.7\, \text{m s}^{-1}$. This is slow enough that the atom can be expected to remain within an exciting laser beam that is 1 cm in diameter for a time sufficiently longer than the effective lifetime $\tau = |2\Delta/\Omega|^2\gamma^{-1}$.

All of these have to be satisfied for a detuning that is large compared with the Rabi frequency. Experimentally acceptable values appear to be $\Delta \approx 2\pi \times 15\, \text{GHz}$ and $\Omega \approx 2\pi \times 300\, \text{MHz}$. Taking into account the necessary size of the Raman interaction region as well, a realistic laser intensity can be achieved around $100\, \text{W cm}^{-2}$. We must use $10\, \text{ns}$, the sodium $3P_{1/2}$ lifetime, for $\tau$. A brief calculation with these parameters gives the exceptionally large values

$$\eta_R \approx 4500 \quad \text{and} \quad K \approx 1200,$$

both orders of magnitude larger than any recorded to date. This seems to give room for an experimental adjustment of the estimated parameters above without compromising the possibility of controlling very high Schmidt number $K$.

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4. Summary and comments

Control of entanglement will be valuable in implementing quantum computing devices. We have reported here an examination of situations in which the Schmidt approach allows straightforward numerical analysis of two-particle entanglement and, probably more valuably, leads to a simple physical interpretation of a control parameter as a linewidth ratio. Thus we connect entanglement control with lifetime control in a quantitative way, via the parameter $\eta$ defined in (3.6).

We have only sketched these consequences for a very simple cross-modular entangled system which consists of an atom and a single photon. There are additional cross-platform instances of beyond-Bell pairing such as electron–photon Schmidt modes, quantum soliton pairs in optical fibres and waveguides, etc., including phased arrays with a large particle number. We argue that retaining focus on high-dimensional contexts will be challenging and valuable.

A specific focus of this investigation was to use control to produce unusually high degree of entanglement values. Examples were noted in several versions of spontaneous emission, including Raman scattering. We showed, in the context of a proposed experiment, that modification of effective radiative lifetimes appeared able to enhance the $K$ value from the previously known range $K = 1–10$ to values three orders of magnitude higher.

Finally, we comment that one obvious consequence of $K > 1000$, say, is that no single eigenvalue $\lambda_n$ can differ very much from any other non-zero eigenvalue. If these eigenvalues are similar enough to permit, at least approximately, an average $\lambda$ to be removed from the Schmidt sum, then we have the following interesting result for the entangled $C(k, q)$ amplitude:

$$C(k, q) = \sum_n \sqrt{\lambda_n} u_n(k)v_n(q)$$

$$\approx \sqrt{\lambda} \sum_n u_n(k)v_n(q) \to \sqrt{\lambda} \delta(q + k). \quad (4.1)$$

The final step, in which the summation is represented by a momentum-conserving delta function, requires an infinite number of states, implying a high-entanglement limit in which $K \to \infty$. Incidentally, the final form is reminiscent of the amplitude of the famous state introduced into physics by Einstein et al. (1935). The consequences of this will have to be pursued elsewhere.

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