Fermi Project Update

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Contents

1 Introduction

Goal: make the probe range and velocity normally distributed.
Method: Add a local variable called technology to the program, use random walks to have technology be normally distributed around the technological capabilities of the abiogenesis seed. Then make probe range and velocity functions of this new technology variable.

2 Technology

2.1 Ornstein-Uhlenbeck (OU) Process

We use an Ornstein-Uhlenbeck[1] process to calculate the random walk for technology:

$$\boxed{dx_t = \theta(\mu - x_t)dt + \sigma dW_t} \tag{1}$$

μ is the location that everything will eventually end up at. If $\mu = 0$, then their is an restoring force towards the origin, that increases with increased distance from the origin. The higher θ is, the quicker μ is reached.[2]

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s \tag{2}$$

[1] https://en.wikipedia.org/wiki/Ornstein-Uhlenbeck_process
[2] We have used the change of variables: $s = e^{2\theta t}$

x_0 is the initial location. It follows that the first moment, or mean of x_t is given by:

$$E[x(t)] = \overline{x}(t) = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) \tag{3}$$

It can similarly be shown that:

$$Cov[x(t), x(s)] = \frac{\sigma^2}{2\theta}\left(e^{-\theta|t-s|} - e^{-\theta|t+s|}\right) \tag{4}$$

It follows that:

$$Var[x(t)] = Cov[x(t), x(t)] = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}) \tag{5}$$

2.1.1 Long-Term Gaussian Behavior ($t >> 1$)

$$\boxed{E[x(t >> 1)] = \mu} \tag{6}$$

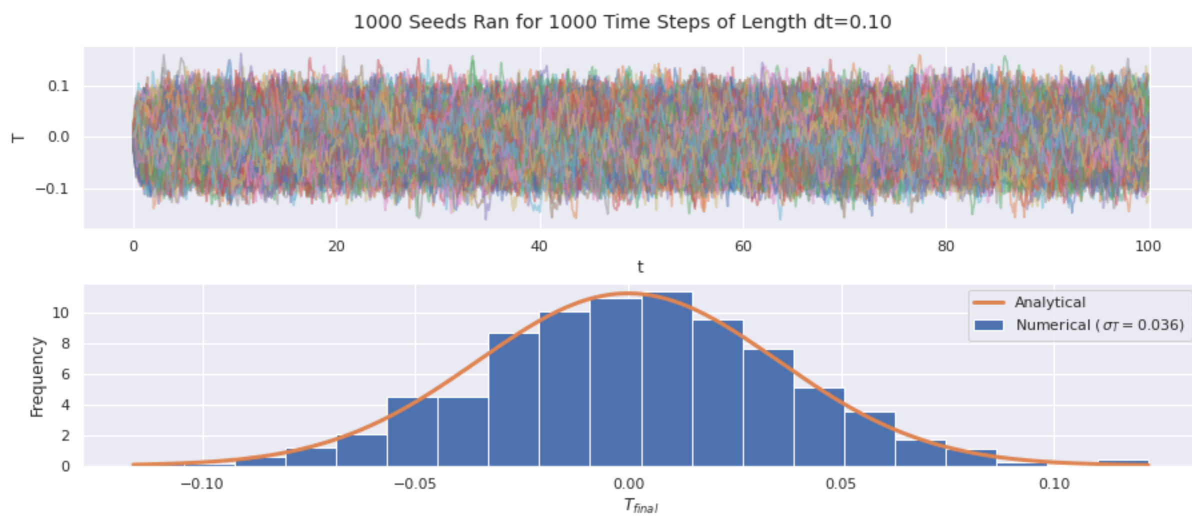
$$\boxed{Var[x(t >> 1)] = \frac{\sigma^2}{2\theta}} \tag{7}$$

2.2 Euler-Maruyama Method

We can use a numerical method called the Euler-Maruyama method to simulate the OU process.

```
num_seeds = 1000
ti = 0              #initial time
tf = 100            #final time
N  = 1000           #number of time steps
#Technology (T)
iTech    = 0       #value to start from
mu       = 0       #value to converge to
sigma    = 0.05    #randomness; directly proportional to width of gaussian
theta    = 1       #rate of convergence; inversely proportional to width of gaussian
#Initialize
dt = float(tf-ti)/N #length of each time step
timeList = np.arange(ti, tf, dt)
techList = np.zeros(N)
techList[0] = iTech #all seeds have the same initial values
fTechList = [] #final value of technology
#Calculate Numerically
fig, (ax1,ax2) = plt.subplots(figsize=(15,5.5), nrows=2, ncols=1)
for _ in range(num_seeds):
    for i in range(1, len(timeList)):                  #i starts at 1
        t = (i-1) * dt                                 #current time
        tech = techList[i-1]                           #current technology value
        dW = np.random.normal(0, np.sqrt(dt))          #Gaussian Noise (mean=0, stdev=sqrt(dt))
        push  = sigma * dW                             #displacing force
        pull  = theta*(mu-tech)*dt                     #restoring force
        techList[i] = tech + push + pull               #next position
    fTechList.append(techList[-1])                     #array keeping track of final values
#Plot Each Run
    ax1.plot(timeList, techList, alpha=.5)
    ax1.set(xlabel='t', ylabel='T')
#Calculate Analytically using a Guassian
mean  = mu  #mean value for large t
stdev = sigma/np.sqrt(2*theta) #width as a normal gaussain
t = np.linspace(min(fTechList), max(fTechList),n)
```

```
techListAnalytic = gaussian(t, mu= mean, sigma=stdev)
#Plot Full Output
ax2.hist(fTechList, bins=20,density=True, label=fr'Numerical ($\,\sigma_T = {np.std(fTechList):.3f}$)')
#Numerically Calculated
ax2.plot(t, techListAnalytic, label=fr'Analytical', linewidth=3) # Analytically Calculated
ax2.set(xlabel=r'$T_{final}$', ylabel='Frequency')
ax2.legend();
fig.suptitle(f"{num_seeds} Seeds Ran for {N} Time Steps of Length dt={dt:.2f}");
fig.subplots_adjust(top=0.92, hspace=.3)
fig.savefig("OUProcess.png")
```



3 Probe Velocity ($v(T)$) as Function of Technology (T)

So general idea here is that total (specific) kinetic energy will scale with technology, then solving for velocity will give $v(T)$. Since the units of specific kinetic energy are the same as velocity squared, we can write the constant of proportionality in terms of the (approximate) initial probe velocity. I now make the following assumptions:

1. Technology starts from $T = 0$
2. When $T = 0$, the initial probe velocity is $v(0) \equiv v_0$ and $v_0 << c$. Thus, when technology is zero, kinetic energy is purely non-relativistic ($K = \frac{1}{2}mv^2$).
3. Specific kinetic energy scales exponentially with technology.[3]

With these assumptions, I can write the scaling relation like:

$$\boxed{Specific\ Kinetic\ Energy \equiv K/m \propto e^{2T} = \left(\frac{v_0^2}{2}\right) e^{2T}} \tag{8}$$

3.1 Non-Relatavistic Kinetic Energy

In this case, kinetic energy is given by:

$$K = \frac{1}{2}mv^2$$

[3]the factor of two helps clean up the solutions

Thus, specific Kinetic Energy is:

$$K/m = \frac{v^2}{2} = \left(\frac{v_0^2}{2}\right) e^{2T}$$

Solving for v and taking the positive solution yields our equation for non-relativstic probe velocity as a function of technology:

$$\boxed{v(T) = v_0\, e^T}$$

3.2 Relativistic Kinetic Energy

In this case, kinetic energy is given by:

$$K = m_0 c^2 \left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right) \tag{9}$$

Thus, specific relativistic kinetic energy is:

$$K/m_0 = c^2 \left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right) = \left(\frac{v_0^2}{2}\right) e^{2T} \tag{10}$$

Solving for v and taking the positive solution yields our equation for non-relativistic probe velocity as a function of technology:

$$\boxed{v(T) = \frac{c v_0 e^T \sqrt{4c^2 + \left(v_0 e^T\right)^2}}{2c^2 + \left(v_0 e^T\right)^2}} \tag{11}$$

- $v(T) = Probe\ Velocity$
- $T = Technology$
- $c = Speed\ of\ Light = 2.99 * 10^{10}\ cm/s$
- v_0=*Approximate Initial Probe Velocity, assuming an initially non-relativistic kinetic energy* (ie: $v(0) = v_0$ iff $v_0 << c$)

4 Probe Range ($r(T)$) as Function of Technology (T)

The expression we are after in this section is a function for the **proper probe range as measured from the planet which is sending out the probe**, which we can call $r(T)$. We want this to be a function of the probes velocity (calculated in the previous section) and the initial probe range, which we can denote r_0.

For a person on the probe, the distance to their destination is contracted as the probes velocity approaches the speed of light; thus we call this contracted probe range as measured from the probe $R(T)$. If we call the probes velocity v and we define:

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \tag{12}$$

then it follows that the **contracted probe range as measured by someone on the probe** is given by this equation for relativistic length contraction:

$$R(T) = \frac{r(T)}{\gamma} = Contracted\ Probe\ Range \tag{13}$$

If we are on the probe, the maximum time we can stay alive in this probe is $t_0 = r_0/v_0$ and the distance to our next destination is given by the contracted probe length $R(T)$, thus it follows that we will calculate our probes velocity to be:

$$v(T) = \frac{R(T)}{t_0} = \frac{r(t)}{\gamma t_0} \tag{14}$$

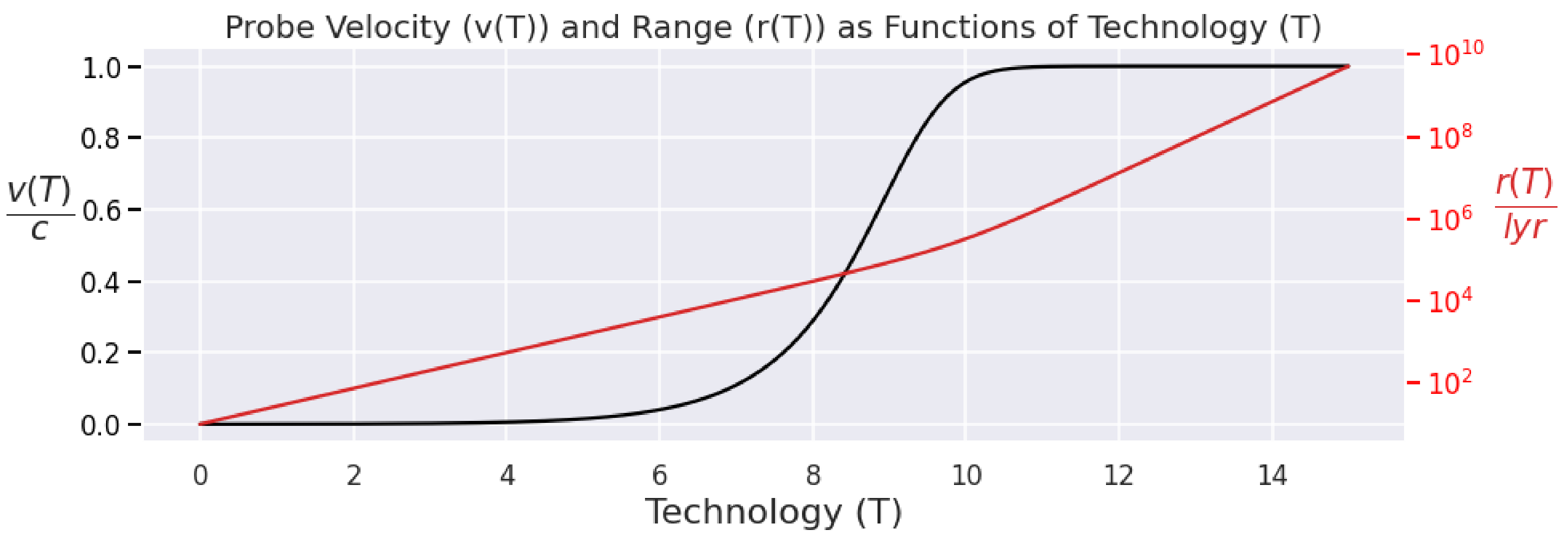
Finally, solving for the proper probe range $r(T)$ yields our desired expression:

$$\boxed{r(T) = \gamma t_0 v(T) = \frac{v(T)t_0}{\sqrt{1-\left(\frac{v(T)}{c}\right)^2}}} \tag{15}$$

- $v(T)$ is the probes velocity as shown in equation 11
- t_0 is the maximum time a probe can stay in space (ie: probes lifetime)

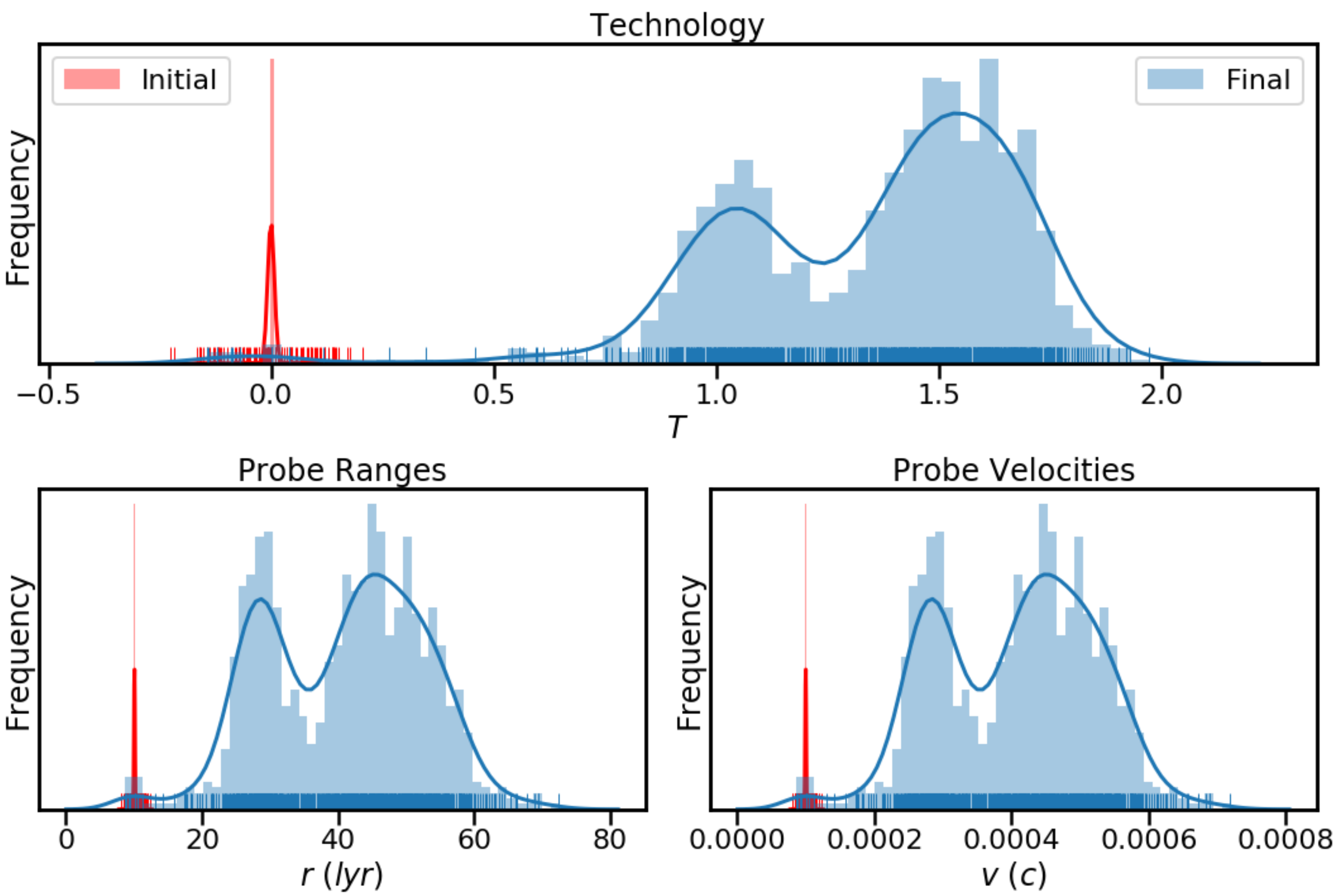
5 Example: $v_0 = 10^{-4}c \approx 30km/s,\ r_0 = 10lyr$

If we assume that initially, when $T = 0$, that probes have velocities approximately 0.01% the speed of light and ranges around $10lyr$, then it follows that the maximum time that a probe can stay in space is 100 thousand years ($t_0 = r_0/v_0 = 100,000yr$). Additionally, for simplicity I let $c = yr = 1$. The plots for $v(T)$ and $r(T)$ are shown below. (Note: probe range on the right hand side is plotted on a log scale)



5.1 Results: Ran For 1 Billion years

- $\theta = 0.1 =$ *Rate of Convergence; inversely proportional to the Gaussian's width*
  - $1/\theta = 10$ steps to converge back to the mean
- $\sigma = 0.9 =$ *Degree of Randomness; directly proportional to the Gaussian's width*
- $v_0 = 10^{-4}c \approx 30km/s =$ *Probe's Initial Velocity*
- $r_0 = 10lyr =$ *Probe's Initial Range*



6 Next Steps

1. Add relativistic corrections to the probe intercept time functionality.
2. Setup three breeds (thin disk, thick disk, halo stars), assign each different constant values[4]
3. Determine what additional parameters we want to have normally distributed.

[4] I can make a function that assigns r_0/v_0 based on galactocentric radius.