Consider the Klein-Gordon equation in the presence of a static Coulomb barrier (minimal coupling),
\[
\left( \Box + m^2 + 2ie\phi \frac{\partial}{\partial t} - e^2 \phi^2 \right) \Psi(x) = 0,
\]
where
\[
\phi = \begin{cases} 
0 & \text{if } z \leq 0, \\
\phi_0 & \text{if } z \geq 0.
\end{cases}
\]
Minimal coupling states:
\[
-i \nabla \rightarrow -i \nabla + eA
\]
\[
i \frac{\partial}{\partial t} \rightarrow i \frac{\partial}{\partial t} - e\phi
\]
\[
\rho(x) \text{ for } z \geq 0 \text{ becomes:}
\]
\[
\rho := \frac{i}{2m} \left[ \Psi^* \frac{\partial}{\partial t} \Psi - \Psi \frac{\partial}{\partial t} \Psi^* + 2ie\phi \Psi^* \Psi \right],
\]
For \( z \leq 0 \) we have:
\[
\Psi_I(z,t) = e^{-i(\omega t - kz)} + Re^{-i(\omega t + kz)}
\]
and
\[
\rho_I = \frac{i}{2m} (\Psi_I^* \frac{\partial}{\partial t} \Psi_I - \Psi_I \frac{\partial}{\partial t} \Psi_I^*)
\]
\[
= \frac{i}{2m} (-i\omega \Psi_I^* \Psi_I - i\omega \Psi_I \Psi_I^*) = \frac{\omega}{m} |\Psi_I|^2 \geq 0
\]
For \( z \geq 0 \) we have:
\[
\Psi_{II}(z,t) = Te^{-i(\omega t - k'z)}
\]
and
\[
\rho_{II} = \frac{i}{2m} (\Psi_{II}^* \frac{\partial}{\partial t} \Psi_{II} - \Psi_{II} \frac{\partial}{\partial t} \Psi_{II}^*) + 2ie\phi \Psi_{II}^* \Psi_{II})
\]
\[
= \frac{i}{2m} (-i\omega |T|^2 - i\omega |T|^2 + 2ie\phi_0 |T|^2) = \frac{|T|^2}{m} (\omega - e\phi_0) \leq 0
\]
since \( e\phi_0 > \omega + m \Rightarrow \omega - e\phi_0 < 0 \).
Using the fact that the set that includes the identity and the three Pauli matrices is a complete basis set for the $2 \times 2$ matrix space, we can expand any arbitrary $2 \times 2$ matrix $A$ as

$$A = \sum_{0}^{3} \alpha_{i} \sigma_{i} \quad ; \quad \sigma_{i} = 1, 1 \hat{\sigma}_{1}, \hat{\sigma}_{2}, \hat{\sigma}_{3}, \quad (10)$$

where $\alpha_{i}$ are C-numbers.

### 2.1

Assume there exists a $2 \times 2$ matrix $A$ that commutes with all three Pauli matrices:

$$[A, \sigma_{i}] = 0 \quad ; \quad i = 0, 1, 2, 3. \quad (11)$$

Using the expansion (10) we can express this condition as: (recall that the Pauli matrices obey the cyclic commutation relation $[\hat{\sigma}_{i}, \hat{\sigma}_{j}] = 2 i \epsilon_{ijk} \hat{\sigma}_{k}$)

$$[A, \hat{\sigma}_{1}] = \sum_{0}^{3} \alpha_{i} [\sigma_{i}, \hat{\sigma}_{1}] = -2i \alpha_{2} \hat{\sigma}_{3} + 2i \alpha_{3} \hat{\sigma}_{2} = 0, \quad (12a)$$

$$[A, \hat{\sigma}_{2}] = \sum_{0}^{3} \alpha_{i} [\sigma_{i}, \hat{\sigma}_{2}] = +2i \alpha_{1} \hat{\sigma}_{3} - 2i \alpha_{3} \hat{\sigma}_{1} = 0, \quad (12b)$$

$$[A, \hat{\sigma}_{3}] = \sum_{0}^{3} \alpha_{i} [\sigma_{i}, \hat{\sigma}_{3}] = -2i \alpha_{1} \hat{\sigma}_{2} + 2i \alpha_{2} \hat{\sigma}_{1} = 0. \quad (12c)$$

From these it is immediately obvious$^{1}$ that, for the commutation requirement (11) to hold, the expansion coefficients involved in the above expressions have to vanish:

$$\alpha_{i} = 0 \quad ; \quad i = 1, 2, 3. \quad (13)$$

We are therefore left with:

$$A = \alpha_{0} \mathbf{1}. \quad (14)$$

### 2.2

Assume there exists a $2 \times 2$ matrix $B$ that anti-commutes with all three Pauli matrices:

$$[B, \sigma_{i}] = 0 \quad ; \quad i = 1, 2, 3. \quad (15)$$

Following the same procedure as in the previous section, we express this condition as: (recall that the anti-commutator of Pauli matrices is twice the identity: $[\hat{\sigma}_{i}, \hat{\sigma}_{j}] = 2 \hat{\sigma}^{2} = 2 \cdot \mathbf{1}$)

$$[B, \hat{\sigma}_{1}] = 2 \beta_{0} \hat{\sigma}_{1} + 2 \beta_{1} \mathbf{1} = 0 \quad \rightarrow \quad \beta_{0} = \beta_{1} = 0, \quad (16a)$$

$$[B, \hat{\sigma}_{2}] = 2 \beta_{0} \hat{\sigma}_{2} + 2 \beta_{2} \mathbf{1} = 0 \quad \rightarrow \quad \beta_{0} = \beta_{2} = 0, \quad (16b)$$

$$[B, \hat{\sigma}_{3}] = 2 \beta_{0} \hat{\sigma}_{3} + 2 \beta_{3} \mathbf{1} = 0 \quad \rightarrow \quad \beta_{0} = \beta_{3} = 0. \quad (16c)$$

$^{1}$Since the Pauli matrices are orthogonal to each-other, there exists no C-number $\alpha$ such that $\alpha \sigma_{i} = \sigma_{j}$.
Therefore only the null matrix anti-commutes with all three Pauli matrices and, as a result, finding four anti-commuting $2 \times 2$ matrices is impossible.

3 \hspace{1cm} \text{(help with \LaTeX{} from Andreas Liapis)}

3.1

We wish to obtain plane wave solutions to the Dirac equation with an arbitrary wavevector; We begin with the Dirac equation in the coordinate representation,

$$(i\partial - m) \Psi = 0,$$  \hspace{1cm} (17)

for which we postulate solutions of the form

$$\Psi(x) = u(k)e^{-k\cdot x}. \hspace{1cm} (18)$$

The eigenvalues of (17) are obtained from:

$$\det(\gamma^0 k_0 - \gamma^i k_i - m) = 0 \Rightarrow k_0 = \pm \sqrt{k^2 + m^2} = \omega \pm$$ \hspace{1cm} (19)

where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \hspace{1cm} (20)$$

Note that

$$\gamma \cdot k = \begin{pmatrix} 0 & \sigma \cdot k \\ -\sigma \cdot k & 0 \end{pmatrix}. \hspace{1cm} (21)$$

The amplitude of the plane wave must obey

$$(k - m) u(k) = 0. \hspace{1cm} (22)$$

We can write equation (22) as

$$\begin{pmatrix} 1(k_0 - m) & -\sigma \cdot k \\ \sigma \cdot k & -1(k_0 + m) \end{pmatrix} u(k) = 0. \hspace{1cm} (23)$$

We now express the four-component vector $u(k)$ as a pair of two-component vectors $\tilde{u}$ and $\tilde{v}$.

$$u(k) = \begin{pmatrix} \tilde{u}(k) \\ \tilde{v}(k) \end{pmatrix} \hspace{1cm} (24)$$

Equation (23) is then reduced to a pair of equations relating these two-component vectors. If we choose the positive energy solution $k_0 = \omega_+$, we find

$$\begin{pmatrix} 1(\omega_+ - m) & -\sigma \cdot k \\ \sigma \cdot k & -1(\omega_+ + m) \end{pmatrix} \begin{pmatrix} \tilde{u}(k) \\ \tilde{v}(k) \end{pmatrix} = 0. \hspace{1cm} (25)$$

In this case, $\tilde{u}$ is the independent quantity, so we wish to solve for $\tilde{v}$. From the second equation, we find that

$$\tilde{v}(k) = \frac{\sigma \cdot k}{\omega_+ + m} \tilde{u}(k) = \frac{\sigma \cdot k}{\omega + m} \tilde{u}(k). \hspace{1cm} (26)$$
Similarly, if we chose the negative energy solution, \( k_0 = \omega_- \) we would find
\[
\begin{pmatrix}
1(\omega_- - m) & -\sigma \cdot k \\
\sigma \cdot k & -1(\omega_- + m)
\end{pmatrix}
\begin{pmatrix}
\tilde{u}(k) \\
\tilde{v}(k)
\end{pmatrix} = 0.
\]
(27)
This time, it is \( \tilde{v} \) that is the independent quantity, and solving the first equation gives:
\[
\tilde{u}(k) = \frac{\sigma \cdot k}{\omega_- - m} \tilde{v}(k) = -\frac{\sigma \cdot k}{\omega + m} \tilde{v}(k).
\]
(28)

3.2

We wish to confirm that the states obtained above are eigenstates of the Hamiltonian
\[
H = \alpha \cdot p + \beta m.
\]
(29)
Consider the positive energy solution first.
\[
H \Psi = \alpha \cdot i \nabla \Psi + \beta m \Psi
\]
(30a)
\[
= \begin{pmatrix}
0 & \sigma \\
-\sigma & 0
\end{pmatrix} \cdot i(ik) \Psi + \begin{pmatrix}
m1 & 0 \\
0 & -m1
\end{pmatrix} \Psi
\]
(30b)
\[
= \begin{pmatrix}
m1 & -\sigma \cdot k \\
\sigma \cdot k & -m1
\end{pmatrix} \Psi
\]
(30c)
\[
= \begin{pmatrix}
m1 & -\sigma \cdot k \\
\sigma \cdot k & -m1
\end{pmatrix} \begin{pmatrix}
\tilde{u} \\
\tilde{v}
\end{pmatrix} e^{ik \cdot x}
\]
(30d)
\[
= \begin{pmatrix}
m1 - \sigma \cdot k \sigma \cdot k \\
\sigma \cdot k - m1\omega + m
\end{pmatrix} \tilde{u} e^{ik \cdot x}
\]
(30e)
\[
= \begin{pmatrix}
m1 - k^2 \frac{1}{\omega + m} \\
\sigma \cdot k(1 - m \frac{1}{\omega + m})
\end{pmatrix} \tilde{u} e^{ik \cdot x}
\]
(30f)
\[
= \begin{pmatrix}
\frac{1}{\sigma \cdot k \omega + m} \\
\tilde{v} \frac{\sigma \cdot k}{\omega + m}
\end{pmatrix} \tilde{u} e^{ik \cdot x}
\]
(30g)
\[
= \omega \begin{pmatrix}
\tilde{u} \\
\tilde{v}
\end{pmatrix} e^{ik \cdot x}
\]
(30h)
Here we have made use of the fact that \( k^2 = \omega^2 - m^2 \) as well as the identity\(^2\) \((\sigma \cdot k)^2 = k^2 + i\sigma \cdot (k \times k) = k^2\). We see that \( \Psi_+ \) is indeed an eigenstate of \( H \) with eigenvalue \( \omega \). For the negative energy solution we start from equation (30c):
\[
H \Psi = \begin{pmatrix}
m1 & -\sigma \cdot k \\
\sigma \cdot k & -m1
\end{pmatrix} \begin{pmatrix}
\frac{\sigma \cdot k}{\omega + m} \tilde{v} \\
\tilde{v}
\end{pmatrix} e^{ik \cdot x}
\]
(31a)
\[
= \begin{pmatrix}
-m \frac{\sigma \cdot k}{\omega + m} - \sigma \cdot k \\
\sigma \cdot k \frac{\sigma \cdot k}{\omega + m} - m1
\end{pmatrix} \tilde{v} e^{ik \cdot x}
\]
(31b)
\[
= \begin{pmatrix}
-m \frac{\sigma \cdot k}{\omega + m} - \sigma \cdot k \\
k^2 \frac{1}{\omega + m} - m1
\end{pmatrix} \tilde{v} e^{ik \cdot x}
\]
(31c)
\[
= -\omega \begin{pmatrix}
-\frac{\sigma \cdot k}{\omega + m} \\
\tilde{v}
\end{pmatrix} e^{ik \cdot x}
\]
(31d)
\(^2\)This can easily be derived from the commutation relation of the Pauli matrices: \( \hat{\sigma}_i \hat{\sigma}_j = i \delta_{ij} + \epsilon_{ijk} k \hat{\sigma}_k \)
We therefore confirm that $\Psi_-$ is an eigenstate of $H$ with eigenvalue $-\omega$. 