Relativistic Quantum Mechanics
Homework 1 (solution)

September 23, 2007

1

Consider the Klein-Gordon equation in the presence of a static Coulomb barrier (minimal coupling),

$$\left(\Box + m^2 + 2ie\phi\frac{\partial}{\partial t} - e^2\phi^2\right)\Psi(x) = 0, \tag{1}$$

where

$$\phi = \begin{cases} 0 & \text{if } z \leq 0, \\ \phi_0 & \text{if } z \geq 0. \end{cases} \tag{2}$$

Minimal coupling states:

$$-i\overrightarrow{\nabla} \rightarrow -i\overrightarrow{\nabla} + e\mathbf{A} \tag{3}$$

$$i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} - e\phi \tag{4}$$

$\rho(x)$ for $z \geq 0$ becomes:

$$\rho := \frac{i}{2m}\left[\Psi^*\frac{\partial}{\partial t}\Psi - \Psi\frac{\partial}{\partial t}\Psi^* + 2ie\phi\Psi^*\Psi\right], \tag{5}$$

For $z \leq 0$ we have:

$$\Psi_I(z,t) = e^{-i(\omega t - kz)} + Re^{-i(\omega t + kz)} \tag{6}$$

and

$$\begin{aligned} \rho_I &= \frac{i}{2m}(\Psi_I^*\frac{\partial\Psi_I}{\partial t} - \Psi_I\frac{\partial\Psi_I^*}{\partial t}) \\ &= \frac{i}{2m}(-i\omega\Psi_I^*\Psi_I - i\omega\Psi_I\Psi_I^*) = \frac{\omega}{m}|\Psi_I|^2 \geq 0 \end{aligned} \tag{7}$$

For $z \geq 0$ we have:

$$\Psi_{II}(z,t) = Te^{-i(\omega t - k'z)} \tag{8}$$

and

$$\begin{aligned} \rho_{II} &= \frac{i}{2m}(\Psi_{II}^*\frac{\partial\Psi_{II}}{\partial t} - \Psi_{II}\frac{\partial\Psi_{II}^*}{\partial t} + 2ie\phi\Psi_{II}^*\Psi_{II}) \\ &= \frac{i}{2m}(-i\omega|T|^2 - i\omega|T|^2 + 2ie\phi_0|T|^2) = \frac{|T|^2}{m}(\omega - e\phi_0) \leq 0 \end{aligned} \tag{9}$$

since $e\phi_0 > \omega + m \overset{m \geq 0}{\Rightarrow} \omega - e\phi_0 < 0$.

2 (TEXing by Andreas Liapis)

Using the fact that the set that includes the identity and the three Pauli matrices is a complete basis set for the 2×2 matrix space, we can expand any arbitrary 2×2 matrix A as

$$A = \sum_0^3 \alpha_i \sigma_i \quad ; \quad \sigma_i = \mathbf{1}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3, \tag{10}$$

where α_i are C-numbers.

2.1

Assume there exists a 2×2 matrix A that commutes with all three Pauli matrices:

$$[A, \sigma_i] = 0 \qquad ; \qquad i = 0, 1, 2, 3. \tag{11}$$

Using the expansion (10) we can express this condition as: (recall that the Pauli matrices obey the cyclic commutation relation $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$)

$$[A, \hat{\sigma}_1] = \sum_0^3 \alpha_i [\sigma_i, \hat{\sigma}_1] = -2i\alpha_2\hat{\sigma}_3 + 2i\alpha_3\hat{\sigma}_2 = 0, \tag{12a}$$

$$[A, \hat{\sigma}_2] = \sum_0^3 \alpha_i [\sigma_i, \hat{\sigma}_2] = +2i\alpha_1\hat{\sigma}_3 - 2i\alpha_3\hat{\sigma}_1 = 0, \tag{12b}$$

$$[A, \hat{\sigma}_3] = \sum_0^3 \alpha_i [\sigma_i, \hat{\sigma}_3] = -2i\alpha_1\hat{\sigma}_2 + 2i\alpha_2\hat{\sigma}_1 = 0. \tag{12c}$$

From these it is immediately obvious[1] that, for the commutation requirement (11) to hold, the expansion coefficients involved in the above expressions have to vanish:

$$\alpha_i = 0 \quad ; \quad i = 1, 2, 3. \tag{13}$$

We are therefore left with:

$$A = \alpha_0 \mathbf{1}. \tag{14}$$

2.2

Assume there exists a 2×2 matrix B that anti-commutes with all three Pauli matrices:

$$[B, \hat{\sigma}_i]_+ = 0 \qquad ; \qquad i = 1, 2, 3. \tag{15}$$

Following the same procedure as in the previous section, we express this condition as: (recall that the anti-commutator of Pauli matrices is twice the identity: $[\hat{\sigma}_i, \hat{\sigma}_i]_+ = 2\hat{\sigma}_i^2 = 2 \cdot \mathbf{1}$)

$$[B, \hat{\sigma}_1] = 2\beta_0\hat{\sigma}_1 + 2\beta_1\mathbf{1} = 0 \quad \rightarrow \quad \beta_0 = \beta_1 = 0, \tag{16a}$$

$$[B, \hat{\sigma}_2] = 2\beta_0\hat{\sigma}_2 + 2\beta_2\mathbf{1} = 0 \quad \rightarrow \quad \beta_0 = \beta_2 = 0, \tag{16b}$$

$$[B, \hat{\sigma}_3] = 2\beta_0\hat{\sigma}_3 + 2\beta_3\mathbf{1} = 0 \quad \rightarrow \quad \beta_0 = \beta_3 = 0. \tag{16c}$$

[1] Since the Pauli matrices are orthogonal to each-other, there exists no C-number α such that $\alpha\sigma_i = \sigma_j$.

Therefore only the null matrix anti-commutes with all three Pauli matrices and, as a result, finding four anti-commuting 2×2 matrices is impossible.

3 (help with TEXing from Andreas Liapis)

3.1

We wish to obtain plane wave solutions to the Dirac equation with an arbitrary wavevector; We begin with the Dirac equation in the coordinate representation,

$$(i\partial\!\!\!/ - m)\,\Psi = 0, \tag{17}$$

for which we postulate solutions of the form

$$\Psi(x) = u(k)e^{-k\cdot x}. \tag{18}$$

The eigenvalues of (17) are obtained from:

$$det(\gamma^0 k_0 - \gamma^i k_i - m) = 0 \Rightarrow k_0 = \pm\sqrt{\mathbf{k}^2 + m^2} = \omega_\pm \tag{19}$$

where

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad ; \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{20}$$

Note that

$$\boldsymbol{\gamma}\cdot\mathbf{k} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{k} \\ -\boldsymbol{\sigma}\cdot\mathbf{k} & 0 \end{pmatrix}. \tag{21}$$

The amplitude of the plane wave must obey

$$(k\!\!\!/ - m)\,u(k) = 0. \tag{22}$$

We can write equation (22) as

$$\begin{pmatrix} \mathbf{1}(k_0 - m) & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -\mathbf{1}(k_0 + m) \end{pmatrix} u(k) = 0. \tag{23}$$

We now express the four-component vector $u(k)$ as a pair of two-component vectors $\tilde{u}$ and $\tilde{v}$.

$$u(k) = \begin{pmatrix} \tilde{u}(k) \\ \tilde{v}(k) \end{pmatrix} \tag{24}$$

Equation (23) is then reduced to a pair of equations relating these two-component vectors. If we choose the positive energy solution $k_0 = \omega_+$, we find

$$\begin{pmatrix} \mathbf{1}(\omega_+ - m) & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -\mathbf{1}(\omega_+ + m) \end{pmatrix} \begin{pmatrix} \tilde{u}(k) \\ \tilde{v}(k) \end{pmatrix} = 0. \tag{25}$$

In this case, $\tilde{u}$ is the independent quantity, so we wish to solve for $\tilde{v}$. From the second equation, we find that

$$\tilde{v}(k) = \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_+ + m}\tilde{u}(k) = \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega + m}\tilde{u}(k). \tag{26}$$

Similarly, if we chose the negative energy solution, $k_0 = \omega_-$ we would find

$$\begin{pmatrix} \mathbf{1}(\omega_- - m) & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -\mathbf{1}(\omega_- + m) \end{pmatrix} \begin{pmatrix} \tilde{u}(k) \\ \tilde{v}(k) \end{pmatrix} = 0. \tag{27}$$

This time, it is $\tilde{v}$ that is the independent quantity, and solving the first equation gives:

$$\tilde{u}(k) = \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega_- - m}\tilde{v}(k) = -\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega + m}\tilde{v}(k). \tag{28}$$

3.2

We wish to confirm that the states obtained above are eigenstates of the Hamiltonian

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m. \tag{29}$$

Consider the positive energy solution first.

$$\begin{aligned}
H\Psi &= \boldsymbol{\alpha}\cdot i\overrightarrow{\nabla}\Psi + \beta m\Psi && (30a)\\
&= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}\cdot i(i\mathbf{k})\Psi + \begin{pmatrix} m\mathbf{1} & 0 \\ 0 & -m\mathbf{1} \end{pmatrix}\Psi && (30b)\\
&= \begin{pmatrix} m\mathbf{1} & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -m\mathbf{1} \end{pmatrix}\Psi && (30c)\\
&= \begin{pmatrix} m\mathbf{1} & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -m\mathbf{1} \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{u}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} \end{pmatrix} e^{ik\cdot x} && (30d)\\
&= \begin{pmatrix} m\mathbf{1} - \boldsymbol{\sigma}\cdot\mathbf{k}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} \\ \boldsymbol{\sigma}\cdot\mathbf{k} - m\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} \end{pmatrix}\tilde{u}e^{ik\cdot x} && (30e)\\
&= \begin{pmatrix} m\mathbf{1} - \mathbf{k}^2\frac{1}{\omega+m} \\ \boldsymbol{\sigma}\cdot\mathbf{k}(1 - m\frac{1}{\omega+m}) \end{pmatrix}\tilde{u}e^{ik\cdot x} && (30f)\\
&= \begin{pmatrix} \mathbf{1}\omega \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m}\omega \end{pmatrix}\tilde{u}e^{ik\cdot x} && (30g)\\
&= \omega\begin{pmatrix} \tilde{u} \\ \tilde{u}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} \end{pmatrix} e^{ik\cdot x} && (30h)
\end{aligned}$$

Here we have made use of the fact that $\mathbf{k}^2 = \omega^2 - m^2$ as well as the identity[2] $(\boldsymbol{\sigma}\cdot\mathbf{k})^2 = \mathbf{k}^2 + i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}) = \mathbf{k}^2$. We see that Ψ_+ is indeed an eigenstate of H with eigenvalue ω. For the negative energy solution we start from equation (30c):

$$\begin{aligned}
H\Psi &= \begin{pmatrix} m\mathbf{1} & -\boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k} & -m\mathbf{1} \end{pmatrix}\begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m}\tilde{v} \\ \tilde{v} \end{pmatrix} e^{ik\cdot x} && (31a)\\
&= \begin{pmatrix} -m\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} - \boldsymbol{\sigma}\cdot\mathbf{k} \\ \boldsymbol{\sigma}\cdot\mathbf{k}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} - m\mathbf{1} \end{pmatrix}\tilde{v}e^{ik\cdot x} && (31b)\\
&= \begin{pmatrix} -m\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m} - \boldsymbol{\sigma}\cdot\mathbf{k} \\ \mathbf{k}^2\frac{1}{\omega+m} - m\mathbf{1} \end{pmatrix}\tilde{v}e^{ik\cdot x} && (31c)\\
&= -\omega\begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\omega+m}\tilde{v} \\ \tilde{v} \end{pmatrix} e^{ik\cdot x} && (31d)
\end{aligned}$$

[2] This can easily be derived from the commutation relation of the Pauli matrices: $\hat{\sigma}_i\hat{\sigma}_j = \mathbf{1}\delta_{ij} + \varepsilon_{ijk}i\hat{\sigma}_k$

We therefore confirm that Ψ_- is an eigenstate of H with eigenvalue $-\omega$.