Relativistic Quantum Mechanics
Homework 10 (solution)

December 3, 2007

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1.1

From the equation of motion for a harmonic oscillator ($\frac{d^2x}{dt^2} = -\omega^2 x$) we have

$$x(t) = x(0)cos(\omega t) + \frac{p(0)}{\omega} sin(\omega t)$$

and

$$\{x(t), x(0)\} = \{p(0), x(0)\} \frac{sin(\omega t)}{\omega} = -\frac{sin(\omega t)}{\omega} \tag{1}$$

Now let's calculate the Schwinger function G(t) which satisfies.

$$\begin{aligned} (\frac{d^2}{dt^2} + \omega^2)G(t) &= -\delta(t) &(2)\\ G(t) &= G^*(t) &(3)\\ G(-t) &= -G(t) &(4) \end{aligned}$$

$$\begin{aligned} G(t) &= \int \frac{d\omega'}{\sqrt{2\pi}} e^{-i\omega' t} \tilde{G}(\omega') &(5)\\ (\frac{d^2}{dt^2} + \omega^2)G(t) &= \frac{1}{\sqrt{2\pi}} \int d\omega' e^{-i\omega' t} (-\omega'^2 + \omega^2) \tilde{G}(\omega') \\ &= -\frac{1}{2\pi} \int d\omega' e^{-i\omega' t} \\ \Rightarrow \tilde{G}(\omega') &= \frac{1}{\sqrt{2\pi}} \frac{1}{\omega'^2 - \omega^2} \\ \Rightarrow G(t) &= \frac{1}{2\pi} \int \frac{d\omega' e^{-i\omega' t}}{\omega'^2 - \omega^2} &(6) \end{aligned}$$

Doing the contour integration one finds

$$G(t) = \frac{sin(\omega t)}{w} \tag{7}$$

So we see that

$$\{x(t), x(0)\} = -G(t) \tag{8}$$

1.2

In the free particle case we have

$$\frac{d^2x}{dt^2} = 0,$$

$$x(t) = x(0) + p(0)t$$

and

$$\{x(t), x(0)\} = -t \tag{9}$$

G(t) in the limit that $\omega \to 0$ is also

$$\begin{aligned} \lim_{\omega\to 0} \frac{sin(\omega t)}{\omega} &= \lim_{\omega\to 0} \frac{tcos(\omega t)}{1} = t \\ \Rightarrow G(t) &= t \end{aligned} \tag{10}$$

2

2.1

In the Heisenberg picture

$$[a^H(\overrightarrow{k}), a^H(\overrightarrow{k}')] = 0 = [a^{\dagger H}(\overrightarrow{k}), a^{\dagger H}(\overrightarrow{k}')] \tag{11}$$

$$[a^H(\overrightarrow{k}), a^{\dagger H}(\overrightarrow{k}')] = \delta^3(\overrightarrow{k} - \overrightarrow{k}') \tag{12}$$

In a new picture

$$a^N(\overrightarrow{k}) = S^{-1}a^H(\overrightarrow{k})S \quad , \quad a^{\dagger N}(\overrightarrow{k}) = S^{-1}a^{\dagger H}(\overrightarrow{k})S \tag{13}$$

$$\begin{aligned} [a^N(\overrightarrow{k}), a^N(\overrightarrow{k}')] &= S^{-1}a^H(\overrightarrow{k})SS^{-1}a^H(\overrightarrow{k}')S - S^{-1}a^H(\overrightarrow{k}')SS^{-1}a^H(\overrightarrow{k})S \\ &= S[a^H(\overrightarrow{k}), a^H(\overrightarrow{k}')]S \\ &= 0 \end{aligned} \tag{14}$$

Similarly,

$$[a^{\dagger N}(\overrightarrow{k}), a^{\dagger N}(\overrightarrow{k}')] = 0 \tag{15}$$

$$[a^N(\overrightarrow{k}), a^{\dagger N}(\overrightarrow{k}')] = \delta^3(\overrightarrow{k} - \overrightarrow{k}') \tag{16}$$

2.2

$$\begin{aligned} [a^S, H_0^S] &= \int d^3k E_{k'} [a^S(\mathbf{k}), a^{S\dagger}(\mathbf{k}')a^S(\mathbf{k}')] \\ &= \int d^3k E_{k'} \{[a^S(\mathbf{k}), a^{S\dagger}(\mathbf{k}')]a^S(\mathbf{k}') + a^{S\dagger}(\mathbf{k}')[a^S(\mathbf{k}), a^S(\mathbf{k}')]\} \\ &= \int d^3k E_{k'} \delta^3(k - k')a^S(\mathbf{k}') \\ &= E_k a^S(\mathbf{k}) \end{aligned} \tag{17}$$

$$\begin{aligned} a^S H_0^S &= H_0^S a^S - [H_0^S, a^S] \\ &= H_0^S a^S + E_k a^S \end{aligned} \tag{18}$$

$$\begin{aligned} a^S (H_0^S)^n &= a^S H_0^S (H_0^S)^{n-1} \\ &= (H_0^S + E_k) a^S (H_0^S)^{n-1} \\ &= (H_0^S + E_k) a^S H_0^S (H_0^S)^{n-2} \\ &= \ldots (induction) \\ &= (H_0^S + E_k)^n a^S \end{aligned} \tag{19}$$

$$\begin{aligned} a^{IP} &= e^{iH_0^S t} a^S e^{-iH_0^S t} \\ &= e^{iH_0^S t} a^S \sum_n \frac{(-it)^n}{n!} (H_0^S)^n \\ &= e^{iH_0^S t} \sum_n \frac{(-it)^n}{n!} a^S (H_0^S)^n \\ &= e^{iH_0^S t} \sum_n \frac{(-it)^n}{n!} (H_0^S + E_k)^n a^S \\ &= e^{iH_0^S t} e^{-it(H_0^S + E_k)} a^S \\ &= e^{-itE_k} a^S \end{aligned} \tag{20}$$

3

3.1

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} (e^{-ikx} a(\mathbf{k}) + e^{ikx} b^\dagger(\mathbf{k})) \tag{21}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} (e^{-ikx} b(\mathbf{k}) + e^{ikx} a^\dagger(\mathbf{k})) \tag{22}$$

$$[a(\mathbf{k}), a(\mathbf{k'})] = [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k'})] = [a(\mathbf{k}), b(\mathbf{k'})] = 0 \tag{23}$$

$$[b(\mathbf{k}), b(\mathbf{k'})] = [b^\dagger(\mathbf{k}), b^\dagger(\mathbf{k'})] = [a^\dagger(\mathbf{k}), b^\dagger(\mathbf{k'})] = 0 \tag{24}$$

$$[a(\mathbf{k}), b^\dagger(\mathbf{k'})] = [b(\mathbf{k}), a^\dagger(\mathbf{k'})] = 0 \tag{25}$$

$$[a(\mathbf{k}), a^\dagger(\mathbf{k'})] = [b(\mathbf{k}), b^\dagger(\mathbf{k'})] = \delta^3(k - k') \tag{26}$$

$$T(\phi(x)\phi(y)) = \Theta(x^0 - y^0)\phi(x)\phi(y) + \Theta(y^0 - x^0)\phi(y)\phi(x) \tag{27}$$

and

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = 0 \tag{28}$$

$$\langle 0|T(\phi^\dagger(x)\phi^\dagger(y))|0\rangle = 0 \tag{29}$$

$$
\begin{aligned}
&\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \\
&\langle 0|\int \frac{d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}}\{\Theta(x^0-y^0)(e^{-ikx}a(\mathbf{k})+e^{ikx}b^\dagger(\mathbf{k}))(e^{-ik'y}b(\mathbf{k'})+e^{ik'y}a^\dagger(\mathbf{k'})) \\
&+\Theta(y^0-x^0)(e^{-ik'y}b(\mathbf{k'})+e^{ik'y}a^\dagger(\mathbf{k'}))(e^{-ikx}a(\mathbf{k})+e^{ikx}b^\dagger(\mathbf{k}))\}|0\rangle \\
= \;& \langle 0|\int \frac{d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}}\{\Theta(x^0-y^0)(e^{-ikx+ik'y}a(\mathbf{k})a^\dagger(\mathbf{k'})) \\
&+\Theta(y^0-x^0)(e^{ikx-ik'y}b(\mathbf{k'})b^\dagger(\mathbf{k}))\}|0\rangle \\
= \;& \langle 0|\int \frac{d^3k}{(2\pi)^3 2k^0}\{\Theta(x^0-y^0)e^{ik(y-x)}+\Theta(y^0-x^0)e^{ik(x-y)}\}|0\rangle \\
= \;& -i\langle 0|\Theta(x^0-y^0)G^+(x-y)-\Theta(y^0-x^0)G^-(x-y)|0\rangle \\
= \;& \langle 0|iG_F(x-y)|0\rangle
\end{aligned} \tag{30}
$$

Similarly

$$
\langle 0|T(\phi^\dagger(x)\phi(y))|0\rangle = \langle 0|iG_F(x-y)|0\rangle \tag{31}
$$

3.2

$$
\begin{aligned}
Q \;=\; & \int d^3x J^0(x) = i\int d^x(\phi^\dagger(x)\dot\phi(x)-\dot\phi^\dagger(x)\phi(x)) \\
= \;& i\int \frac{d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}}(e^{ikx}a^\dagger(\mathbf{k})+e^{-ikx}b(\mathbf{k}))(-ik'^0 e^{-ik'x}a(\mathbf{k'})+ik'^0 e^{ik'x}b^\dagger(\mathbf{k'})) \\
&-(-ik'^0 e^{-ik'x}b(\mathbf{k'})+ik'^0 e^{ik'x}a^\dagger(\mathbf{k'}))(e^{-ikx}a(\mathbf{k})+e^{ikx}b^\dagger(\mathbf{k'})) \\
= \;& i\int \frac{d^3k d^3k'}{(2\pi)^3 2\sqrt{k^0 k'^0}} \\
&(-ik'^0 a(\mathbf{k'})b(\mathbf{k})e^{-ix(k+k')}+ik'^0 a^\dagger(\mathbf{k})b^\dagger(\mathbf{k'})e^{ix(k+k')} \\
&+ik'^0 b(\mathbf{k})b^\dagger(\mathbf{k'})e^{-ix(k-k')}-ik'^0 a^\dagger(\mathbf{k'})a(\mathbf{k'})e^{ix(k-k')}) \\
&-(-ik'^0 b(\mathbf{k'})a(\mathbf{k})e^{-ix(k+k')}+ik'^0 b^\dagger(\mathbf{k})a^\dagger(\mathbf{k'})e^{ix(k+k')} \\
&+ik'^0 a^\dagger(\mathbf{k'})a(\mathbf{k})e^{-ix(k-k')}-ik'^0 b(\mathbf{k'})b^\dagger(\mathbf{k'})e^{ix(k-k')}) \\
= \;& i\int \frac{d^3k d^3k'}{2\sqrt{k^0 k'^0}} \\
&\{-ik'^0 a(\mathbf{k'})b(\mathbf{k})+ik'^0 a^\dagger(\mathbf{k})b^\dagger(\mathbf{k'})+ik'^0 b(\mathbf{k'})a(\mathbf{k})-ik'^0 b^\dagger(\mathbf{k})a^\dagger(\mathbf{k'})\}\delta(k+k') \\
&+\{ik'^0 b(\mathbf{k})b^\dagger(\mathbf{k'})-ik'^0 a^\dagger(\mathbf{k})a(\mathbf{k'})-ik'^0 a^\dagger(\mathbf{k'})a(\mathbf{k})+ik'^0 b(\mathbf{k'})b^\dagger(\mathbf{k})\}\delta(k-k') \\
= \;& i\int \frac{d^3k}{2k^0} \\
&i\{k^0 a(-\mathbf{k})b(\mathbf{k})-k^0 a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})-k^0 b(-\mathbf{k})a(\mathbf{k})+k^0 b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})\}+ \\
&i\{k^0 b(\mathbf{k})b^\dagger(\mathbf{k})-k^0 a^\dagger(\mathbf{k})a(\mathbf{k})-k^0 a^\dagger(\mathbf{k})a(\mathbf{k})+k^0 b(\mathbf{k})b^\dagger(\mathbf{k})\} \\
= \;& \int d^3k(a^\dagger(\mathbf{k})a(\mathbf{k})-b(\mathbf{k})b^\dagger(\mathbf{k}))
\end{aligned} \tag{32}
$$