Relativistic Quantum Mechanics
Homework 3 (solution)

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1.1

$$S_3\, u_+(p) = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} \tilde{u}(p) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{u}(p) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \pm\tilde{u}(p) \\ \sigma_3\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{u}(p) \end{pmatrix} \neq const.\, u_+(p) \tag{1}$$

So $u_+(p)$ is an eigenstate of S_3 ($\tilde{u}(p)$ is an eigenstate of σ_3) if it commutes with $\vec{\sigma}\cdot\vec{p}$ or equivalently if $p_1 = p_2 = p_3 = 0$ (or when $\mathbf{p} = 0$).

$$\begin{aligned}[][\sigma_3, \vec{\sigma}\cdot\vec{p}] &= \sigma_3\,\vec{\sigma}\cdot\vec{p} - \vec{\sigma}\cdot\vec{p}\,\sigma_3 = 0 \Leftrightarrow \\ & [\sigma_3, \sigma_i]\,p_i + \sigma_i\,[\sigma_3, p_i] = 0 \Leftrightarrow \\ & p_1 = p_2 = p_3 = 0 \end{aligned} \tag{2}$$

Similarly for the negative energy solutions.

1.2

$$\begin{aligned} u_+(p,h) &= \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \tilde{u}(p,h) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{u}(p,h) \end{pmatrix} \\ u_-(p,h) &= \sqrt{\frac{E+m}{2m}}\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{v}(p,h) \\ \tilde{v}(p,h) \end{pmatrix} \end{aligned} \tag{3}$$

so that

$$\begin{aligned} \overline{u}_+(p)\,u_+(p) &= 1 \\ \overline{u}_-(p)\,u_-(p) &= -1 \end{aligned} \tag{4}$$

with

$$\begin{aligned} h(p)\,\tilde{u}(p,\pm\frac{1}{2}) &= \pm\frac{1}{2}\,\tilde{u}(p,\pm\frac{1}{2}) \qquad (5)\\ \Rightarrow \frac{\overrightarrow{s}\cdot\overrightarrow{p}}{|p|}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \pm\frac{1}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \Rightarrow \begin{pmatrix} p_3 & p_1 - ip_2\\ p_1 + ip_2 & -p_3\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \pm|p|\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad (6)\end{aligned}$$

$$\Rightarrow \begin{aligned} (p_3 \mp |p|)\alpha + (p_1 - ip_2)\beta &= 0\\ (p_1 + ip_2)\alpha - (p_3 \pm |p|)\beta &= 0 \end{aligned} \qquad (7)$$

For $\alpha = 1$ we then have $\beta = \frac{p_1+ip_2}{p_3\pm|p|}$, so $\tilde{u}(p,\pm\frac{1}{2}) = a_1\begin{pmatrix}1\\ \frac{p_1+ip_2}{p_3\pm|p|}\end{pmatrix}$. And a_1 is given by

$$\begin{aligned} \tilde{u}^{\dagger}_{\pm}\tilde{u}_{\pm} &= 1 \Rightarrow\\ a_1 &= \sqrt{\frac{|p|\pm p_3}{2|p|}} \end{aligned} \qquad (8)$$

and

$$u_+(p,\pm\frac{1}{2}) = \sqrt{\frac{E+m}{2m}}\cdot\sqrt{\frac{|p|\pm p_3}{2|p|}}\begin{pmatrix} 1\\ \frac{p_1+ip_2}{p_3\pm|p|}\\ \frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\\ \frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\frac{p_1+ip_2}{p_3\pm|p|}\end{pmatrix}$$

$$u_+(p,\pm\frac{1}{2}) = \frac{1}{2}\sqrt{\frac{(E+m)(|p|\pm p_3)}{m|p|}}\begin{pmatrix} 1\\ \frac{p_1+ip_2}{p_3\pm|p|}\\ \frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\\ \frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\frac{p_1+ip_2}{p_3\pm|p|}\end{pmatrix} \qquad (9)$$

Similarly one finds

$$v_-(p,\pm\frac{1}{2}) = \frac{1}{2}\sqrt{\frac{(E+m)(|p|\pm p_3)}{m|p|}}\begin{pmatrix} -\frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\\ -\frac{\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m}\frac{p_1+ip_2}{p_3\pm|p|}\\ 1\\ \frac{p_1+ip_2}{p_3\pm|p|}\end{pmatrix} \qquad (10)$$

2

We have

$$
\begin{aligned}
\mathbf{x} &= x(t,x) \\
\mathbf{x^2} &= \langle \mathbf{x}|\mathbf{x}\rangle = t^2 - x^2 \\
|\mathbf{x}\rangle &= t\,|e_t\rangle + x\,|e_x\rangle \\
\langle \mathbf{x}|\mathbf{x}\rangle &= t^2\,\langle e_t|e_t\rangle + t\,x\,\langle e_t|e_x\rangle + x\,t\,\langle e_x|e_t\rangle + x^2\,\langle e_x|e_x\rangle \\
&= t^2 - x^2 \qquad (11)
\end{aligned}
$$

From Eq.11 we have

$$
\begin{aligned}
\langle e_t|e_t\rangle &= 1 \qquad (12)\\
\langle e_x|e_x\rangle &= -1 \qquad (13)\\
\langle e_t|e_x\rangle &= \langle e_x|e_t\rangle = 0 \qquad (14)
\end{aligned}
$$

Therefore the completeness relation can be written as

$$
|e_t\rangle\langle e_t| - |e_x\rangle\langle e_x| = 1 \qquad (15)
$$

So that

$$
\begin{aligned}
\mathbf{1}\cdot\mathbf{v} &= (|e_t\rangle\langle e_t| - |e_x\rangle\langle e_x|)\cdot(v_1\,|e_t\rangle + v_2\,|e_x\rangle) \\
&= |e_t\rangle\,\langle e_t|v_1|e_t\rangle + |e_t\rangle\,\langle e_t|v_2|e_x\rangle - |e_x\rangle\,\langle e_x|v_1|e_t\rangle - |e_x\rangle\,\langle e_x|v_x|e_t\rangle \\
&= v_1\,|e_t\rangle + v_2|e_x\rangle = \mathbf{v} \qquad (16)
\end{aligned}
$$

3

3.1

$$
\begin{aligned}
x^\mu &\rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \\
\psi(x) &\rightarrow \psi'(x') = S(\Lambda)\psi(x) \qquad (17)
\end{aligned}
$$

Dirac equation becomes

$$
(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \rightarrow (i\gamma^\mu(\partial_\mu)' - m)\psi'(x') = 0 \qquad (18)
$$

$$
\begin{aligned}
(i\gamma^\mu\Lambda^\nu{}_\mu\partial_\nu - m)S\;\psi(x) &= 0 \\
\Rightarrow (i\Lambda^\nu{}_\mu S^{-1}\gamma^\mu S\partial^\nu - m)\psi(x) &= 0 \\
\overset{S^{-1}\gamma^\mu S=\Lambda^\mu{}_\nu\gamma^\nu}{\Rightarrow}\quad (i\gamma^\nu\partial_\nu - m)\psi(x) &= 0 \qquad (19)
\end{aligned}
$$

where γ^μ transforms under parity as follows

$$
\begin{aligned}
S^{-1}\gamma^0 S &= \gamma^0 \\
S^{-1}\overrightarrow{\gamma}S &= -\overrightarrow{\gamma}
\end{aligned}
\qquad (20)
$$

Since

$$\begin{aligned}(\gamma^0)^{-1}\gamma^0\gamma^0 &= \gamma^0 \\ (\gamma^0)^{-1}\vec{\gamma}\gamma^0 &= -\vec{\gamma}\end{aligned} \tag{21}$$

We can see that one possibility is that $S = \pm\gamma^0$. From Eq.19 we see that the Dirac equation is invariant under a parity transformation.

3.2

$$\begin{aligned}\psi(x) \to \psi'(x^0, -\vec{x}) &= S\psi(x) = \pm\gamma^0\psi(x) \\ \overline{\psi}(x) \to \overline{\psi}'(x^0, -\vec{x}) &= \overline{\psi}(x)S^{-1} = \pm\overline{\psi}(x)\gamma^0\end{aligned}$$

$$\begin{aligned}\overline{\psi}\psi &\to \overline{\psi}S^{-1}S\psi = \overline{\psi}\psi \quad \text{(scalar)} \\ \overline{\psi}\gamma_5\psi &\to \overline{\psi}S^{-1}\gamma_5 S\psi = \overline{\psi}\gamma^0\gamma_5\gamma^0\psi = -\overline{\psi}\gamma^0\gamma^0\gamma_5\psi = -\overline{\psi}\gamma_5\psi \quad \text{(pseudoscalar)} \\ \overline{\psi}\gamma^\mu\psi &\to \overline{\psi}\gamma^0\gamma^\mu\gamma^0\psi = \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi \quad \text{(vector)} \\ \overline{\psi}\gamma_5\gamma^\mu\psi &\to \overline{\psi}S^{-1}\gamma_5\gamma^\mu S\psi = \overline{\psi}S^{-1}\gamma_5 SS^{-1}\gamma^\mu S\psi = -\Lambda^\mu{}_\nu\overline{\psi}\gamma_5\gamma^\nu\psi \quad \text{(pseudovector)} \\ \overline{\psi}\sigma^{\mu\nu}\psi &\to \overline{\psi}S^{-1}(\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu))S\psi \\ &= \frac{i}{2}\overline{\psi}[S^{-1}\gamma^\mu SS^{-1}\gamma^\nu SS^{-1}\gamma^\nu SS^{-1}\gamma^\mu S]\psi \\ &= \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\overline{\psi}\sigma^{\rho\sigma}\psi \qquad \text{(tensor)}\end{aligned}$$

3.3

$$u'_+ = \gamma^0 \begin{pmatrix} \tilde{u} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{u} \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \frac{-\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{u} \end{pmatrix} \tag{22}$$

$$u'_- = \gamma^0 \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{v} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\tilde{v} \\ -\tilde{v} \end{pmatrix} \tag{23}$$