Relativistic Quantum Mechanics
Homework 4 (solution)

October 22, 2007

1

We are looking for a unitary transformation (U, V) such that

$$\gamma^\mu_M = U^\dagger \gamma^\mu_P U \tag{1}$$

and

$$\gamma^\mu_W = V^\dagger \gamma^\mu_P V \tag{2}$$

With $U^\dagger = U^{-1}, V^\dagger = V^{-1}$.
Dirac basis:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\gamma^0_M = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \gamma^0\gamma^2 = U^{-1}\gamma^0 U \tag{3}$$

$$\Rightarrow \boxed{U\gamma^2 = -\gamma^0 U \gamma^0} \tag{4}$$

$$\gamma^1_M = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} = -\gamma^1\gamma^2 = U^{-1}\gamma^1 U \tag{5}$$

$$\Rightarrow U\gamma^2\gamma^1 = \gamma^1 U \Rightarrow U\gamma^0\gamma^1 = -\gamma^0\gamma^1 U$$

$$\Rightarrow \boxed{\{U, \gamma^0\gamma^1\} = 0} \tag{6}$$

$$\gamma^2_M = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = -\gamma_2 = U^{-1}\gamma^2 U \tag{7}$$

$$\Rightarrow -U\gamma^2 = \gamma^2 U \Rightarrow \boxed{\{U, \gamma^2\} = 0} \tag{8}$$

$$\gamma^3_M = \begin{pmatrix} \imath\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} = \gamma^2\gamma^3 = U^{-1}\gamma^3 U \tag{9}$$

$$\Rightarrow \boxed{\{U, \gamma^0\gamma^3\} = 0}, \boxed{[U, \gamma^1\gamma^3]} \tag{10}$$

U can be written as a linear combination of Dirac matrices Γ^α.
1, γ^2 don't anti-commute with γ^2, $\gamma_5\gamma^\mu$ does when $\mu = 0$ and $\sigma^{\mu\nu}$ when $\nu = 0$

so we are left with 8 matrices $\gamma^{\mu\neq 0}$, γ_5, $\gamma_5\gamma^2$, $\sigma^{\mu 2}$.
Out of them only γ^2 and σ^{02} satisfy the remaining conditions, therefore U can be written as:

$$U = c_1\gamma^0 + c_2\gamma^0\gamma^2 \tag{11}$$

or

$$U = \frac{1}{\sqrt{2}}\gamma^0(\mathbf{1} + \gamma^2) \tag{12}$$

Weyl representation:

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{13}$$

$$\gamma_W^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \gamma_5 \tag{14}$$

$$\gamma_W^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \gamma^i \tag{15}$$

$$\gamma_W^0 = V^{-1}\gamma^0 V \Rightarrow \boxed{V\gamma_5 = \gamma^0 V} \tag{16}$$

$$\gamma_W^i = V^{-1}\gamma^i V \Rightarrow \boxed{[S, \gamma^i] = 0} \tag{17}$$

Only $\mathbf{1}$ and $\gamma_5\gamma^0$ satisfy the above, so

$$V = c_1\mathbf{1} + c_2\gamma_5\gamma^0 \tag{18}$$

or

$$V = \frac{i}{\sqrt{2}}(\mathbf{1} + \gamma_5\gamma^0) \tag{19}$$

2

2.1

$$|\gamma_5 - \mathbf{1}\lambda| = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 2\lambda^2 + 1 = 0$$
$$\Rightarrow \lambda = \pm 1 \tag{20}$$

γ_5 has doubly degenerate eigenvalues and its eigenstates are:

$$e_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad and \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, e_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{21}$$

2.2

$$P_L = \frac{1}{2}(1-\gamma_5),\ P_R = \frac{1}{2}(1+\gamma_5) \tag{22}$$

$$\begin{aligned} P^2 &= \frac{1}{4}(1\mp\gamma_5)(1\mp\gamma_5) = \frac{1}{4}(1\mp 2\gamma_5+\gamma_5\gamma_5) \\ &= \frac{1}{4}(1\mp 2\gamma_5+1) = \frac{1}{2}(1\mp\gamma_5) = P \end{aligned} \tag{23}$$

2.3

Acting on Dirac Eq.

$$P(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{24}$$

Also

$$\{\gamma_5,\gamma^\mu\} = 0 \Rightarrow \{P,\gamma^\mu\} = \{\frac{1}{2}(1\mp\gamma_5),\gamma^\mu\} = \gamma^\mu \tag{25}$$

So,

$$\begin{aligned} P_L(i\gamma^\mu\partial_\mu - m)(\psi_L(x)+\psi_R(x)) &= 0 \Rightarrow \overset{P_L\psi=\psi_L}{=} i\gamma^\mu\partial_\mu\psi_R - m\psi_L = 0 \\ \Rightarrow \not{p}\psi_R - m\psi_L &= 0 \end{aligned} \tag{26}$$

Similarly

$$\not{p}\psi_L - m\psi_R = 0 \tag{27}$$

The two equations decouple in the ultrarelativistic limit (m$\rightarrow$ 0).

3

Discussion and derivation here (by Hsin-Chia Cheng, University of California).