Relativistic Quantum Mechanics
Homework 5 (solution)

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1.1

$$\psi(x) = \sum_{h_{\pm 1/2}} \int \frac{d^3k}{(2\pi)^{3/2}} (a(k,h)e^{-ik\cdot x}u(k,h) + b^*(k,h)e^{ik\cdot x}v(k,h)) \quad (1)$$

$$\psi^\dagger(x) = \sum_{h_{\pm 1/2}} \int \frac{d^3k}{(2\pi)^{3/2}} (a^*(k,h)e^{ik\cdot x}u^\dagger(k,h) + b(k,h)e^{-ik\cdot x}v^\dagger(k,h)) \quad (2)$$

$$\begin{aligned}\psi^\dagger(x)\psi(x) = \sum_{h,h'_{\pm 1/2}} \int \frac{d^3k\, d^3k'}{(2\pi)^3} \; & [a^*(k',h')a(k,h)e^{i(k'-k)\cdot x}u^\dagger(k',h')u(k,h) + \\ & a^*(k',h')b^*(k,h)e^{i(k'+k)\cdot x}u^\dagger(k',h')v(k,h) + \\ & b(k',h')b^*(k,h)e^{i(k-k')\cdot x}v^\dagger(k',h')v(k,h) + \\ & b(k',h')a(k,h)e^{-i(k+k')\cdot x}v^\dagger(k',h')u(k,h)] \quad (3)\end{aligned}$$

In the massive normalization

$$u^\dagger(k,h)u(k',h') = \frac{k^0}{m}\delta_{k'k}\delta_{h'h} = v^\dagger(k,h)v(k',h') \quad (4)$$

and

$$u^\dagger v = 0 = v^\dagger u \quad (5)$$

So Eq.3 becomes

$$\psi^\dagger(x)\psi(x) = \sum_{h_{\pm 1/2}} \int \frac{d^3k}{(2\pi)^3} (|a(k,h)|^2 + |b(k,h)|^2)\frac{k^0}{m} \quad (6)$$

Requiring that

$$\int d^3x\, \psi^\dagger(x)\psi(x) = 1$$

we get

$$\sum_{h_{\pm 1/2}} \int d^3k\, (|a(k,h)|^2 + |b(k,h)|^2) = \frac{8\pi^3 m}{k^0} \quad (7)$$

1.2

In the massless normalization

$$u^\dagger(k,h)u(k',h') = k^0\delta_{k'k}\delta_{h'h} = v^\dagger(k,h)v(k',h') \tag{8}$$

and we get

$$\sum_{h_{\pm 1/2}} \int d^3k \left(|a(k,h)|^2 + |b(k,h)|^2\right) = \frac{8\pi^3}{k^0} \tag{9}$$

2

2.1

Dirac Hamiltonian in the presence of a constant magnetic field (minimal coupling) becomes

$$H = \overrightarrow{\alpha} \cdot (\overrightarrow{p} - e\overrightarrow{A}) + \beta m \tag{10}$$

Defining

$$\overrightarrow{\Pi} = \overrightarrow{p} - e\overrightarrow{A} \tag{11}$$

we have

$$H = \overrightarrow{\alpha} \cdot \overrightarrow{\Pi} + \beta m \tag{12}$$

and

$$\begin{aligned} [\Pi_i, H] &= [p_i + eA_i, \alpha_j(p_j - eA_i) + \beta m] \\ &= -\alpha_j[p_i + eA_i, p_j - eA_j] \end{aligned} \tag{13}$$

We also see that

$$[\Pi_i, \Pi_j] \overset{[p_i,A_j]\sim\partial_i A_j}{\sim} \epsilon_{ijk}B_k \neq 0 \tag{14}$$

since

$$\begin{aligned} &[p_i - eA_i, p_j - eA_i] = \\ &(p_i - eA_i)(p_j - eA_j) - (p_j - eA_j)(p_i + eA_i) = \\ &p_ip_j - ep_iA_j - eA_ip_j + e^2A_iA_j - p_jp_i + ep_jA_i + eA_jp_i - e^2A_jA_i = \\ &-e[p_i, A_j] + e[p_j, A_i] = ie(\partial_iA_j - \partial_jA_i) = -ie\epsilon_{ijk}B_k \neq 0 \end{aligned} \tag{15}$$

And $\overrightarrow{\Pi}$ isn't conserved (similarly $\overrightarrow{p}$ doesn't commute with H).

Total angular momentum $\overrightarrow{J}$ is conserved

$$
\begin{aligned}
&[J_i, H] = [\epsilon_{ijk} x_j \Pi_k + \frac{1}{2}\tilde{\alpha}_i, H] = \\
&[\epsilon_{ijk} x_j \Pi_k + \frac{1}{2}\tilde{\alpha}_i, \alpha_m \Pi_m + \beta m] = \\
&[\epsilon_{ijk} x_j \Pi_k, \alpha_m \Pi_m] + [\frac{1}{2}\tilde{\alpha}_i, \alpha_m \Pi_m] + [\epsilon_{ijk} x_j \Pi_k, \beta m] + [\frac{1}{2}\tilde{\alpha}_i, \beta m] = \\
&\epsilon_{ijk} x_j \alpha_m [\Pi_k, \Pi_m] + \epsilon_{ijk} \alpha_m [x_j, \Pi_m] \Pi_k + \frac{1}{2}[\tilde{\alpha}_i, \alpha_m] \Pi_m = \\
&\epsilon_{ijk} x_j \alpha_m (-ie) \epsilon_{kml} B_l + i\epsilon_{ijk} \alpha_m \delta_{jm} \Pi_k + i\epsilon_{imn} \alpha_n \Pi_m = \\
&-ie\epsilon_{ijk} \epsilon_{mlk} x_j \alpha_m B_l + i\epsilon_{ijk} \alpha_j \Pi_k + i\epsilon_{imn} \alpha_n \Pi_m = \\
&-ie\epsilon_{ijk} \epsilon_{mlk} x_j \alpha_m B_l + i\epsilon_{ijk} \alpha_j \Pi_k + i\epsilon_{ijk} \alpha_k \Pi_j = \\
&-ie\epsilon_{ijk} \epsilon_{mlk} x_j \alpha_m B_l + i\epsilon_{ijk} \alpha_j \Pi_k - i\epsilon_{ijk} \alpha_j \Pi_k = 0
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
&H^2 \overset{\{\alpha_i, \beta\}=0}{=} \overrightarrow{\Pi}^2 + m^2 \mathbf{1} = \\
&\alpha_i \alpha_j (p_i - eA_i)(p_j - eA_j) + m^2 = \\
&(\frac{1}{2}\{\alpha_i, \alpha_j\} + \frac{1}{2}[\alpha_i, \alpha_j]) \Pi_i \Pi_j + m^2 = \\
&(\delta_{ij} + i\epsilon_{ijk} \alpha_k) \Pi_i \Pi_j + m^2 = \\
&\overrightarrow{\Pi}^2 + i\epsilon_{ijk} \alpha_k \Pi_i \Pi_j + m^2 = \\
&\overrightarrow{\Pi}^2 + i\epsilon_{ijk} \alpha_k \frac{1}{2}[\Pi_i, \Pi_j] + m^2 = \\
&\overrightarrow{\Pi}^2 + \epsilon_{ijk} \alpha_k \frac{1}{2} e\epsilon_{ijm} B_m + m^2 = \\
&\overrightarrow{\Pi}^2 + \alpha_k \delta_{km} B_m + m^2 = \\
&\overrightarrow{\Pi}^2 + \overrightarrow{\alpha} \cdot \overrightarrow{B} + m^2
\end{aligned}
\tag{17}
$$

Since $[H, H^2] = 0$ we see that $\overrightarrow{\Pi}^2 + \overrightarrow{\alpha} \cdot \overrightarrow{B} + m^2$ is a conserved quantity (such as energy and total angular momentum).

2.2

Solving the characteristic equation for the Hamiltonian

$$
\begin{vmatrix} (m-E)\mathbf{1} & \overrightarrow{\sigma} \cdot (\overrightarrow{p} - e\overrightarrow{A}) \\ \overrightarrow{\sigma} \cdot (\overrightarrow{p} - e\overrightarrow{A}) & -(m+E)\mathbf{1} \end{vmatrix} = 0 \tag{18}
$$

or

$$
\begin{aligned}
&(\overrightarrow{\sigma} \cdot (\overrightarrow{p} - e\overrightarrow{A}))(\overrightarrow{\sigma} \cdot (\overrightarrow{p} - e\overrightarrow{A})) = E^2 - m^2 \\
\Rightarrow\ &E^2 - m^2 = [(\overrightarrow{p} - e\overrightarrow{A}) \cdot (\overrightarrow{p} - e\overrightarrow{A}) - e\overrightarrow{\sigma} \cdot \overrightarrow{B}]
\end{aligned}
\tag{19}
$$

3

We need to show that

$$
\overline{u}(p_2)\gamma^\mu u(p_1) = \frac{(p_1 + p_2)^\mu}{2m} \overline{u}(p_2) u(p_1) - \frac{i(p_1 - p_2)_\nu}{2m} \overline{u}(p_2) \sigma^{\mu\nu} u(p_1) \tag{20}
$$

where,

$$(\gamma^\mu p_{1\mu} - m)u(p_1) = 0, \quad \overline{u}(p_2)(\gamma^\mu p_{2\mu} - m) = 0 \tag{21}$$

We start with

$$\begin{aligned}
&\overline{u}(p_2)\sigma^{\mu\nu}(p_1 - p_2)_\nu u(p_1) = \\
&\overline{u}(p_2)i[(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})p_{1\nu} - (\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)p_{2\nu}]u(p_1) = \\
&\overline{u}(p_2)i[\gamma^\mu\gamma^\nu p_{1\nu} - p_1^\mu - p_2^\mu + \gamma^\nu p_{2\nu}\gamma^\mu]u(p_1) = \\
&\overline{u}(p_2)i[\gamma^\mu m - (p_1 + p_2)^\mu + m\gamma^\mu]u(p_1)
\end{aligned} \tag{22}$$

or that

$$-\frac{i}{2m}\overline{u}(p_2)\sigma^{\mu\nu}(p_1 - p_2)_\nu u(p_1) = -\overline{u}(p_2)[\frac{1}{2m}(p_1 + p_2)^\mu - \gamma^\mu]u(p_1) \tag{23}$$

So that RHS of Eq.20 gives

$$RHS = \overline{u}(p_2)\gamma^\mu u(p_1) \tag{24}$$