Relativistic Quantum Mechanics
Homework 6 (solution)

December 4, 2007

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1.1

$$u'(p') = Su(p), \quad S = cosh\frac{\omega}{2} - \alpha_3 sinh\frac{\omega}{2} \tag{1}$$

$$\alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

$$\begin{aligned} Su(p) &= \begin{pmatrix} cosh\frac{\omega}{2} & 0 & -sinh\frac{\omega}{2} & 0 \\ 0 & cosh\frac{\omega}{2} & 0 & sinh\frac{\omega}{2} \\ -sinh\frac{\omega}{2} & 0 & cosh\frac{\omega}{2} & 0 \\ 0 & sinh\frac{\omega}{2} & 0 & cosh\frac{\omega}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} cosh\frac{\omega}{2} \\ 0 \\ -sinh\frac{\omega}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\gamma+1}{2}} \\ 0 \\ -\sqrt{\frac{\gamma-1}{2}} \\ 0 \end{pmatrix} \end{aligned} \tag{3}$$

where

$$cosh\omega = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

.

1.2

Symmetries of $p^\mu = (p, 0, 0, p)$ commute with $P_0 - P_3$.

$$[P_0 - P_3, M_{12}] = 0 \tag{4}$$

$$[P_0 - P_3, \Pi_1] = [P_0 - P_3, M_{01} - M_{13}] = 0 \tag{5}$$

$$[P_0 - P_3, \Pi_2] = [P_0 - P_3, M_{02} - M_{23}] = 0 \tag{6}$$

A rotation on the x-y plane around z-axis leaves $p^\mu = (p, 0, 0, p)$ invariant (eq.4).

$$M_{12} = \pm J_3, \quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & sin\alpha & 0 \\ 0 & -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

Also,

$$\Lambda^{\mu}{}_{\nu} = e^{-\alpha(M_{01}-M_{13})} = e^{-\alpha M_{01}} e^{\alpha M_{13}} e^{\frac{\alpha^2}{2}[M_{01},M_{13}]} \tag{8}$$

or

$$\Lambda^{\mu}{}_{\nu} = e^{-\alpha M_{01}} e^{\alpha M_{13}} e^{-\frac{\alpha^2}{2} M_{03}} =$$

$$= \begin{pmatrix} cosh\alpha & -sinh\alpha & 0 & 0 \\ -sinh\alpha & cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & 0 & sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -sin\alpha & 0 & cos\alpha \end{pmatrix} \begin{pmatrix} cosh\frac{\alpha^2}{2} & 0 & 0 & -sinh\frac{\alpha^2}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -sinh\frac{\alpha^2}{2} & 0 & 0 & cosh\frac{\alpha^2}{2} \end{pmatrix}$$

(first boost along z-axis, then rotation about y-axis, then boost along x-axis)

Similarly,

$$\Lambda^{\mu}{}_{\nu} = e^{-\alpha(M_{02}-M_{23})} = e^{-\alpha M_{02}} e^{\alpha M_{23}} e^{\frac{\alpha^2}{2}[M_{02},M_{23}]} \tag{9}$$

or

$$\Lambda^{\mu}{}_{\nu} = e^{-\alpha M_{02}} e^{\alpha M_{23}} e^{-\frac{\alpha^2}{2} M_{03}} =$$

$$= \begin{pmatrix} cosh\alpha & 0 & -sinh\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -sinh\alpha & 0 & cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\alpha & sin\alpha \\ 0 & 0 & -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} cosh\frac{\alpha^2}{2} & 0 & 0 & -sinh\frac{\alpha^2}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -sinh\frac{\alpha^2}{2} & 0 & 0 & cosh\frac{\alpha^2}{2} \end{pmatrix}$$

(first boost along z-axis, then rotation about x-axis, then boost along y-axis)

2

In $2+1$ D we have

$$x^{\mu} = (t, x^1, x^2), \quad \eta_{\mu\nu} = diag(1, -1, -1) \tag{10}$$

and an infinitesimal transformation δx^{μ} (translations along all axes, boosts along x^1, x^2 and a rotation in the x^1-x^2 plane)

$$\delta x^{\mu} = \epsilon^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} \tag{11}$$

The associated vector field is

$$\hat{R}(\epsilon, \omega) = -(\epsilon^{\mu}\partial_{\mu} + \omega^{\mu\nu} x_{\nu} \partial_{\mu}) \tag{12}$$

There are 6 independent generators; 3 translations (p_{μ}), 2 boosts (M_{0i}) and 1 rotation (M_{12}) with commutation relations

$$[p_{\mu}, M_{\nu\sigma}] = \eta_{\mu\nu} p_{\sigma} - \eta_{\mu\sigma} p_{\mu} \tag{13}$$

$$[p_{\mu}, p_{\nu}] = 0 \tag{14}$$

$p^2 = \eta^{\mu\nu} p_{\mu} p_n u$ is the Casimir of the group

$$[p^2, p_{\mu}] = 0 \tag{15}$$

$$[p^2, M_{\mu\nu}] = 0 \tag{16}$$

3

3.1

The Lagrangian of a free relativistic point particle of mass m is

$$L = -\frac{m}{\gamma} \tag{17}$$

where

$$\gamma = (1 - v^2)^{-1/2}$$

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$$p = \frac{\partial L}{\dot{x}} = \frac{m\dot{x}}{\gamma} \tag{18}$$

and

$$\frac{\partial p}{\partial t} - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{m}{\gamma}\ddot{x} = 0 \tag{19}$$

3.2

not graded

A massless particle has a constant velocity c (or momentum $p = 1/\lambda$) and the Lagrangian would be a constant since its motion is simply a straight line. A massless field instead can be described by

$$L = -\frac{1}{16} F_{\alpha\beta} F^{\alpha\beta} \tag{20}$$

and

$$\frac{\partial L}{\partial \phi} = \partial^{\mu} \frac{\partial L}{\partial(\partial^{\mu}\phi)} \tag{21}$$

3.3

not graded

$$H = p\dot{x} - L = \gamma m, \quad \frac{d\dot{x}}{dt} = -i[\dot{x}, H] = 0 = \ddot{x} \tag{22}$$