Relativistic Quantum Mechanics
Homework 7 (solution)

November 11, 2007

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$$\mathbf{A} = |\frac{1}{2}, \pm\frac{1}{2}\rangle^A \quad (1)$$

$$\mathbf{B} = |\frac{1}{2}, \pm\frac{1}{2}\rangle^B \quad (2)$$

$$\mathbf{J} = \mathbf{A} \oplus \mathbf{B} = |\frac{1}{2}, \pm\frac{1}{2}\rangle^A \oplus |\frac{1}{2}, \pm\frac{1}{2}\rangle^B \quad (3)$$

so adding 2 spin-1/2 states we get

$$\mathbf{J} = (0,1) \quad J_3 = (0,0,1/2,-1/2) \quad (4)$$

$$\mathbf{J} = |0,0\rangle \quad (5)$$

$$\mathbf{J} = |1,(0,-\frac{1}{2},+\frac{1}{2})\rangle \quad (6)$$

$$\mathbf{J}^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2A_3B_3 + A_+B_- + A_-B_+ \quad (7)$$

and

$$\mathbf{J}^2|J\rangle = \hbar^2 J(J+1)|J\rangle \quad (8)$$

$$J_3|J\rangle = \hbar J_3|J\rangle \quad (9)$$

$$\begin{aligned}
\mathbf{J}^2|\frac{1}{2}^A, \frac{1}{2}^B\rangle &= (\mathbf{A}^2 + \mathbf{B}^2 + 2A_3B_3 + A_+B_- + A_-B_+)|\frac{1}{2}^A, \frac{1}{2}^B\rangle \\
&= (\frac{3}{4} + \frac{3}{4})\hbar^2|\frac{1}{2}^A, \frac{1}{2}^B\rangle + \frac{1}{2}\hbar^2|\frac{1}{2}^A, \frac{1}{2}^B\rangle \\
&= 2\hbar^2|\frac{1}{2}^A, \frac{1}{2}^B\rangle \quad (10)
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}^2|\frac{1}{2}^A, -\frac{1}{2}^B\rangle &= (\mathbf{A}^2 + \mathbf{B}^2 + 2A_3B_3 + A_+B_- + A_-B_+)|\frac{1}{2}^A, -\frac{1}{2}^B\rangle \\
&= (\frac{3}{4} + \frac{3}{4})\hbar^2|\frac{1}{2}^A, -\frac{1}{2}^B\rangle - \frac{1}{2}\hbar^2|\frac{1}{2}^A, -\frac{1}{2}^B\rangle + \hbar^2|-\frac{1}{2}^A, \frac{1}{2}^B\rangle \\
&= \hbar^2(|\frac{1}{2}^A, -\frac{1}{2}^B\rangle + |-\frac{1}{2}^A, \frac{1}{2}^B\rangle) \quad (11)
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}^2|-\frac{1}{2}^A,\frac{1}{2}^B\rangle &= (\mathbf{A}^2+\mathbf{B}^2+2A_3B_3+A_+B_-+A_-B_+)|-\frac{1}{2}^A,\frac{1}{2}^B\rangle \\
&= (\frac{3}{4}+\frac{3}{4})\hbar^2|-\frac{1}{2}^A,\frac{1}{2}^B\rangle-\frac{1}{2}\hbar^2|-\frac{1}{2}^A,\frac{1}{2}^B\rangle+\hbar^2|\frac{1}{2}^A,-\frac{1}{2}^B\rangle \\
&= \hbar^2(|\frac{1}{2}^A,-\frac{1}{2}^B\rangle+|-\frac{1}{2}^A,\frac{1}{2}^B\rangle) \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}^2|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle &= (\mathbf{A}^2+\mathbf{B}^2+2A_3B_3+A_+B_-+A_-B_+)|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle \\
&= (\frac{3}{4}+\frac{3}{4})\hbar^2|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle+\frac{1}{2}\hbar^2|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle \\
&= 2\hbar^2|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle \qquad (13)
\end{aligned}$$

In the new basis

$$\begin{aligned}
|\psi_1\rangle &= |\frac{1}{2}^A,\frac{1}{2}^B\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}(|\frac{1}{2}^A,-\frac{1}{2}^B\rangle+|-\frac{1}{2}^A,\frac{1}{2}^B\rangle) \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}(|\frac{1}{2}^A,-\frac{1}{2}^B\rangle-|-\frac{1}{2}^A,\frac{1}{2}^B\rangle) \\
|\psi_4\rangle &= |-\frac{1}{2}^A,-\frac{1}{2}^B\rangle
\end{aligned}$$

$\mathbf{J}^2$, J_3, A_3 and B_3 are diagonal.

$$\begin{aligned}
J_x|\psi_1\rangle &= (\frac{A_++A_-}{2}+\frac{B_++B_-}{2})|\frac{1}{2}^A,\frac{1}{2}^B\rangle \\
&= \frac{\hbar}{2}(|-\frac{1}{2}^A,\frac{1}{2}^B\rangle+|\frac{1}{2}^A,-\frac{1}{2}^B\rangle)=\frac{\hbar}{\sqrt{2}}|\psi_2\rangle \\
J_x|\psi_2\rangle &= (\frac{A_++A_-}{2}+\frac{B_++B_-}{2})\frac{1}{\sqrt{2}}(|-\frac{1}{2}^A,\frac{1}{2}^B\rangle+|\frac{1}{2}^A,-\frac{1}{2}^B\rangle) \\
&= \frac{\hbar}{\sqrt{2}}(|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle+|\frac{1}{2}^A,\frac{1}{2}^B\rangle)=\frac{\hbar}{\sqrt{2}}(|\psi_2\rangle+|\psi_4\rangle) \\
J_x|\psi_3\rangle &= (\frac{A_++A_-}{2}+\frac{B_++B_-}{2})\frac{1}{\sqrt{2}}(|-\frac{1}{2}^A,\frac{1}{2}^B\rangle-|\frac{1}{2}^A,-\frac{1}{2}^B\rangle) \\
&= \frac{\hbar}{2\sqrt{2}}(|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle-|\frac{1}{2}^A,\frac{1}{2}^B\rangle)+\frac{\hbar}{2\sqrt{2}}(|\frac{1}{2}^A,\frac{1}{2}^B\rangle-|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle)=0 \\
J_x|\psi_4\rangle &= (\frac{A_++A_-}{2}+\frac{B_++B_-}{2})|-\frac{1}{2}^A,-\frac{1}{2}^B\rangle \\
&= \frac{\hbar}{2}(|\frac{1}{2}^A,-\frac{1}{2}^B\rangle+|-\frac{1}{2}^A,\frac{1}{2}^B\rangle)=\frac{\hbar}{\sqrt{2}}|\psi_2\rangle
\end{aligned}$$

So in the above basis

$$J_x=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0&0\\1&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix} \qquad (14)$$

Similarly

$$\begin{aligned} J_y|\psi_1\rangle &= \frac{i\hbar}{\sqrt{2}}|\psi_2\rangle \\ J_y|\psi_2\rangle &= \frac{i\hbar}{\sqrt{2}}(-|\psi_1\rangle + |\psi_4\rangle) \\ J_y|\psi_3\rangle &= 0 \\ J_y|\psi_4\rangle &= -\frac{i\hbar}{\sqrt{2}}|\psi_2\rangle \end{aligned}$$

and

$$J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{15}$$

Finally,

$$\begin{aligned} J_z|\psi_1\rangle &= \hbar|\psi_1\rangle \\ J_z|\psi_2\rangle &= 0 \\ J_z|\psi_3\rangle &= 0 \\ J_z|\psi_4\rangle &= -\hbar|\psi_4\rangle \end{aligned}$$

and

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16}$$

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2.1

$$\begin{aligned} J_i &= -\frac{1}{2}\epsilon_{ijk}\sigma^{jk} = -\frac{1}{2}\epsilon_{ijk} i\gamma^j\gamma^k \\ J_i^\dagger &= \frac{1}{2}\epsilon_{ijk} i(\gamma^j\gamma^k)^\dagger \\ &= \frac{1}{2}\epsilon_{ijk} i(\gamma^k)^\dagger(\gamma^j)^\dagger = \\ &= \frac{1}{2}\epsilon_{ijk} i\gamma^k\gamma^j = -\frac{1}{2}\epsilon_{ijk} i\gamma^j\gamma^k \\ &= J_i \end{aligned} \tag{17}$$

$$\begin{aligned} K_i &= \frac{1}{2}\sigma_{0i} = \frac{1}{2}\frac{i}{2}[\gamma^0, \gamma^i] \\ K_i^\dagger &= -\frac{1}{2}\frac{i}{2}(\gamma^0\gamma^i - \gamma^i\gamma^0)^\dagger \\ &= -\frac{1}{2}\frac{i}{2}(-\gamma^i\gamma^0 + \gamma^0\gamma^i) \\ &= -K_i \end{aligned} \tag{18}$$

2.2

We are looking for

$$\gamma_h^\mu = S\gamma^\mu S^{-1} \tag{19}$$

such that

$$(\gamma_h^\mu)^\dagger = \gamma_h^\mu \tag{20}$$

or

$$\begin{aligned} \gamma_h^\mu &= S\gamma^\mu S^{-1} \rightarrow \\ (\gamma_h^\mu)^\dagger &= (S^{-1})^\dagger (\gamma^\mu)^\dagger S^\dagger \\ &= S(\gamma^\mu)^\dagger S^{-1} \neq S\gamma^\mu S^{-1} \end{aligned} \tag{21}$$

which is true for the Dirac representation, so there can be no unitary matrix S that can satisfy Eq. 19 and 20.
Similarly, for the Hamiltonian of the Dirac equation to be hermitian (as any physical system) we must have

$$\beta = \gamma^0 = (\gamma^0)^\dagger, \quad \gamma^i = -(\gamma^i)^\dagger \tag{22}$$

for any representation of the Dirac matrices. Therefore, we can never have $(\gamma^\mu)^\dagger = \gamma^\mu$.

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3.1

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\phi\Box\phi - \frac{m^2}{2}\phi^2 \\ &= -\frac{1}{2}(\partial_\mu(\phi\partial^\mu\phi) - \partial_\mu\phi\partial^\mu\phi) - \frac{m^2}{2}\phi^2 \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{2}\partial_\mu(\phi\partial^\mu\phi) \end{aligned} \tag{23}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \tag{24}$$

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) = \partial_\mu(\partial^\mu\phi) = \Box\phi \tag{25}$$

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_i}^{t_f} d^4x\delta\mathcal{L} \\ &= \int_{t_i}^{t_f} d^4x[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)] \\ &= \int_{t_i}^{t_f} d^4x[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)] \\ &= \int_{t_i}^{t_f} d^4x[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi) - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta\phi] \\ &= \int_{t_i}^{t_f} d^4x(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}))\delta\phi + \int_{t_i}^{t_f} d^3x\frac{\partial\mathcal{L}}{\partial(\partial\dot{\phi})}\delta\phi\Big|_{t_i}^{t_f} \end{aligned} \tag{26}$$

The last term goes to zero so that the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$
$$-m^2\phi - \Box\phi = 0 \Rightarrow (\Box + m^2)\phi = 0 \tag{27}$$

Adding a total divergence term (or a constant) to the Lagrangian doesn't change Euler-Lagrange equations.

3.2

Since $\mathcal{L} = \mathcal{L}(\phi, \Box\phi) = \mathcal{L}(\phi, \partial_\mu\phi)$ (Euler-Lagrange equations remain unchanged) we see that $\mathcal{L}$ really depends on ϕ, $\partial_\mu\phi$ only.