Reconciling Cloning Fidelities

Irfan Ali Khan and John C. Howell
Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

Much literature is available on developing a theoretical framework for Quantum Cloning, however there is no significant bridge between how recent definitions of fidelity in the framework of optical implementations of cloning [1] are related to the theoretical density matrix representation. A simple expression for an experimental measurement of fidelity is proposed in this paper based upon previous successful implementations [1, 2]. It is shown to hold for *all* symmetric N→M QC, with the symmetry requirement being lifted for ancilla-free cloners. The fidelity is verified explicitly for the 1→2 UQC based on stimulated emission proposed in [1], where the indistinguishability of the output clones was also discussed.

Introduction

In 1982 Nick Herbert proposed an interesting method of superluminal communication by attempting to utilize the non-locality of Quantum Mechanics theorized in entanglement between quantum particles [3]. However, apart from being a result of the corner stone of relativity, the upper limit of the speed of communication to that of light in vacuum also spares physicists the trouble of time-travel paradoxes. Hence the solution to Herbert's proposal was quick in coming in the form of the "no cloning theorem" by Wooters and Zurek [4], which states that it is impossible to *perfectly* clone an unknown pure quantum state.

This simple theorem resulted in the conception of quantum cryptography [5–8], and later clarification of the possibility of *imperfect cloning* led to the field of quantum cloning [9–15]. As well as being of central importance to quantum cryptography, cloning has become a topic of interest in the field of quantum computation [16] and in the foundations of quantum physics in general. Barnum *et al* [17] later extended the theory to the "no-broadcasting theorem" to include the impossibility of perfectly cloning mixed quantum states. In doing so the use of fidelity [18] was introduced as a measure of similarity of two quantum states by comparing their respective density matrices.

The first theoretical model of a universal quantum cloner (UQC) was proposed by Bužek and Hillery [10] wherein unknown pure quantum states pointing anywhere on the Bloch sphere were copied equally well. The Hilbert-Schmidt norm was primarily used in that paper as a measure of cloning efficiency, however their UQC was also shown to have a fidelity of 5/6. Since then the use of fidelity has become a popular method for characterizing cloners. The fidelity of 5/6 as an upper bound for $1 \rightarrow 2$ universal quantum copying has since been proved via various methods [12, 13, 19], and indeed the method by Gisin [13] showed this to be the upper bound over which superluminal communication would be possible.

Bruss *et al* carefully characterized the cloning process, and re-derived the Buzek and Hillery cloner as the optimal UQC via constructive proof [19]. In fact various other types of cloners have been proposed and their respective upper limits explored [20–22].

Experimental realizations of quantum cloning came only recently, with De Martini *et al* [23] and Lamas-Linares *et al* [2] demonstrating the first experimental UQC following a proposal to clone the polarization state of photons by Simon *et al* [1]. In the Simon *et al* proposal, however, the two output clones are not in spatially distinguishable modes, which leads to problems in defining fidelity in the popular density matrix formulation (as will be seen later). Hence, to circumvent this problem, the fidelity was re-defined in an intuitively appealing form as "the average of the relative frequency of photons with the correct polarization in the final state".

Indeed this form predicted a fidelity of 5/6 for the proposed cloner, and using the same definition of fidelity Lamas-Linares *et al* measured a fidelity of 5/6 in their experimental realization. Most recently, Fasel *et al* demonstrated an experimental QC via linear amplification, once again using the Simon *et al* definition to measure a fidelity of 5/6 [24].

In this paper we wish to explore the relationship between the two definitions of fidelity and to what extent they are equivalent. In the process we shall re-express the Simon *et al* fidelity in a form which is more conducive to experimental situations. It will be seen that the two definitions are indeed analogous for all symmetric cloners, with the symmetry requirement being lifted for ancilla-free cloners.

Definition of Fidelity

The density matrix of a pure state can be most generally expressed as

$$\varrho_{pure} = |\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + \vec{s}\cdot\vec{\sigma}) \qquad (1)$$

where $\mathbb{1}$ is the identity matrix, $\vec{s}$ is the Bloch vector of the state and $\vec{\sigma}$ are the x, y and z pauli matrices. The generality of this expression can be intuitively verified by observing that the identity gives the density matrix a trace of 1 while the pauli matrices allow the Bloch vector to point anywhere within the Bloch sphere. For simplicity we shall at first talk about only $1 \rightarrow 2$ cloners, later extending our results to the $N \rightarrow M$ case.

As shown in [19], when the conditions of unitarity, symmetry and isotropy are imposed on a UQC, the Bloch vector of the original qubit cannot rotate. It does however shrink by a 'shrinking factor' η, in keeping with the no-cloning theorem

$$\varrho_{red.} = \frac{1}{2}(\mathbb{1} + \eta \vec{s} \cdot \vec{\sigma}) = \eta |\psi\rangle\langle\psi| + \frac{1}{2}(1-\eta)\mathbb{1} \quad (2)$$

where $|\psi\rangle$ is the original input qubit. This reduction in the magnitude of the vector results in a mixed state of the two output clones. The symmetry requirement and the definition for a clone to be as close as possible to the original density matrix resulted in the following theoretical expression for local fidelity [19]

$$F_T = \langle\psi|\varrho_{red.}|\psi\rangle \quad (3)$$

This definition is analogous to the one originally advocated by Schumacher [18] in that it is a measure of the similarity of the original and cloned density matrices.

In the Simon *et al* proposal [1], quantum cloning via stimulated emission was proposed wherein an expression for Fidelity was given based on "right" and "wrong" photons. We shall rewrite the definition in an analogous form which is more instructive in an experimental environment:

$$F_S = \frac{R_{Clones}}{R_{Clones} + R_{Noise}} = Prob(Clone) \quad (4)$$

where R_{Clones} and R_{Noise} are count rates of clones (right photons) and noise (wrong photons) respectively, and Prob(Clone) is the probability of any one count being a clone. One can see that this is exactly equivalent to the expression of fidelity used in Eq. (5) by Lamas-Linares *et al* [2]. What this implies is that in an experimental setting, by measuring the above rates one can easily calculate the fidelity of a cloner without needing to resort to density matrices.

However, to what extent is F_S truly representative of the density matrices of the input and output qubits? In order to answer this question let us first take a look at the example of a 1→2 cloner. In the following example we have a general expression for the output field of a non-optimised universal cloner [19] with an input signal of $|0\rangle$. Here, as well as in the rest of the paper, we shall only consider an input state of $|0\rangle$ in order to avoid redundancy.

$$|\varphi\rangle_{output} = \alpha|00A\rangle + \beta(|01B_1\rangle + |10B_2\rangle) + \gamma|11C\rangle \quad (5)$$

where

$$|\alpha|^2 X_A + |\beta|^2 (X_{B_1} + X_{B_2}) + |\gamma|^2 X_C = 1 \quad (6)$$

due to normalization, $X_A = \langle A|A\rangle$ etc, and $|0\rangle$ and $|1\rangle$ are assumed to be orthonormal. For generality we have lifted the normalization condition of the ancillary states as assumed in [19]. The ancilla states can be expressed in terms of a complete orthonormal basis spanning the D-dimensional Hilbert space of the ancilla:

$$|J\rangle = \sum_{i=1}^{D} \lambda_i^{(J)} |\lambda_i\rangle \quad (7)$$

where J is a general ancilla state. Therefore

$$X_J = \langle J|J\rangle = \sum_{i=1}^{D} |\lambda_i^{(J)}|^2 = Tr\{|J\rangle\langle J|\}_{anc.} \quad (8)$$

where $Tr\{\}_{anc.}$ implies tracing over the ancillary states. The last equality has been added for convenience in the rest of the paper.

It must be kept in mind that the two output qubits are in different modes (distinguishable). In this example the "right" photons of the output are in state $|0\rangle$. Observing the output field in Eq. (5) we note that with probability $|\alpha|^2 X_A$ both output photons are in state $|0\rangle$, with probability $|\beta|^2(X_{B_1} + X_{B_2})$ only one of the output photons is in state $|0\rangle$, and with probability $|\gamma|^2 X_C$ none of the photons are in state $|0\rangle$. Using this in Eq. (4) gives us

$$\begin{aligned} F_S &= \frac{2|\alpha|^2 X_A + |\beta|^2 (X_{B_1} + X_{B_2})}{2|\alpha|^2 X_A + 2|\beta|^2 (X_{B_1} + X_{B_2}) + 2|\gamma|^2 X_C} \\ &= |\alpha|^2 + |\beta|^2 \frac{(X_{B_1} + X_{B_2})}{2} \end{aligned} \quad (9)$$

due to the normalization condition in Eq. (6). Let us now compare this with F_T via the reduced density matrix of one of the clones (by tracing over the ancilla and the other clone):

$$\begin{aligned} \varrho_{red.} = \; & |\alpha|^2 |0\rangle\langle 0| \cdot Tr\{|A\rangle\langle A|\}_{anc.} \\ & + |\gamma|^2 |1\rangle\langle 1| \cdot Tr\{|C\rangle\langle C|\}_{anc.} \\ & + |\beta|^2 |1\rangle\langle 1| \cdot Tr\{|B_2\rangle\langle B_2|\}_{anc.} \\ & + |\beta|^2 |0\rangle\langle 0| \cdot Tr\{|B_1\rangle\langle B_1|\}_{anc.} \\ & + \alpha\beta^* |0\rangle\langle 1| \cdot Tr\{|A\rangle\langle B_2|\}_{anc.} \\ & + \alpha^*\beta |1\rangle\langle 0| \cdot Tr\{|B_2\rangle\langle A|\}_{anc.} \\ & + \gamma^*\beta |1\rangle\langle 0| \cdot Tr\{|B_1\rangle\langle C|\}_{anc.} \\ & + \gamma\beta^* |0\rangle\langle 1| \cdot Tr\{|C\rangle\langle B_1|\}_{anc.} \end{aligned} \quad (10)$$

This gives

$$F_T = \langle 0|\varrho_{red.}|0\rangle = |\alpha|^2 X_A + |\beta|^2 X_{B_1} \quad (11)$$

using Eq. (8). We can see that the only solution for $F_S = F_T$ is $X_{B_1} = X_{B_2}$, i.e. the symmetric cloner. *Thus F_S is a valid measure of fidelity for all symmetric $1 \rightarrow 2$ cloners.* It should be noted that no limits were placed on the dimensionality of the ancilla. It can be seen that this result also extends to *all* $1 \rightarrow 2$ ancilla-free cloners.

N→M cloner

This analysis can be extended to the general N→M cloner. Once again it must be kept in mind that each output clone is distinguishable. Let us assume without loss of generality that the input state of the clone = $|[0]_N\rangle$, i.e. N particles in state $|0\rangle$. In the same vein as above, the output of the cloner will be

$$|\varphi\rangle_{output} = \sum_{i=0}^{M} \beta_i \sum_{k=1}^{\frac{M!}{i!(M-i)!}} |[0]_i[1]_{M-i}\rangle^{(k)} |B_i^{(k)}\rangle \tag{12}$$

where we have summed over all the possible output states. $|[0]_i[1]_{M-i}\rangle^{(k)}$ represents the k'th permutation of an output with 'i' clones in state $|0\rangle$ and 'M-i' clones in state $|1\rangle$. There are $\frac{M!}{i!(M-i)!}$ such distinguishable permutations. For generality, each permutation is associated with a distinct ancilla state $|B_i^{(k)}\rangle$. The states $|0\rangle$ and $|1\rangle$ are assumed to be orthonormal, and the normalization condition is now given by

$$\sum_{i=0}^{M} |\beta_i|^2 \sum_{k=1}^{\frac{M!}{i!(M-i)!}} X_{B_i^{(k)}} = 1 \tag{13}$$

Summing the number of "right" output states as before, the simplistic expression of fidelity gives us

$$\begin{aligned} F_S &= \frac{\sum_{i=0}^{M} i.|\beta_i|^2.\sum_{k=1}^{\frac{M!}{i!(M-i)!}} X_{B_i^{(k)}}}{\sum_{i=0}^{M} M.|\beta_i|^2.\sum_{k=1}^{\frac{M!}{i!(M-i)!}} X_{B_i^{(k)}}} \\ &= \frac{1}{M}\sum_{i=0}^{M} i.|\beta_i|^2.\sum_{k=1}^{\frac{M!}{i!(M-i)!}} X_{B_i^{(k)}} \end{aligned} \tag{14}$$

where the normalization condition in Eq. (13) was used in the second step.

Now, only the diagonal terms of the reduced density matrix contribute to F_T, therefore we shall ignore all off-diagonal elements in calculating the reduced density matrix of the first clone (i.e. the clone in the first output mode). In order to to this we will need to separate out the cases when the first mode is in state $|0\rangle$, and when it is in state $|1\rangle$. Let us consider the case where an output has 'i' clones in state $|0\rangle$ and 'M-i' clones in state $|1\rangle$. Keeping the first mode in state $|0\rangle$, there are $\frac{(M-1)!}{(i-1)!(M-i)!} = s$ ways to distribute the remaining clones in the remaining M-1 modes. We can specify the first 's' permutations to fall within this category, while the remaining permutations have state $|1\rangle$ in the first mode.

This allows us to write

$$\begin{aligned} \varrho_{red.} &= \sum_{i=1}^{M-1} |\beta_i|^2 \{\sum_{k=1}^{s} |0\rangle\langle 0| X_{B_i^{(k)}} + \sum_{k=s+1}^{\frac{M!}{i!(M-i)!}} |1\rangle\langle 1| X_{B_i^{(k)}}\} \\ &+ |\beta_0|^2 |1\rangle\langle 1| X_{B_0^{(1)}} + |\beta_M|^2 |0\rangle\langle 0| X_{B_M^{(1)}} + OffDiag... \end{aligned} \tag{15}$$

which gives us

$$\begin{aligned} F_T &= |\beta_M|^2 X_{B_M^{(1)}} + \sum_{i=1}^{M-1} |\beta_i|^2 \sum_{k=1}^{\frac{(M-1)!}{(M-i)!(i-1)!}} X_{B_i^{(k)}} \\ &= \sum_{i=1}^{M} |\beta_i|^2 \sum_{k=1}^{\frac{(M-1)!}{(M-i)!(i-1)!}} X_{B_i^{(k)}} \end{aligned} \tag{16}$$

Comparing this to Eq. (14), we see that the only solution to $F_S = F_T$ is $X_{B_i^{(a)}} = X_{B_i^{(b)}}$ for all a, b. *Therefore the two expressions for fidelity are equivalent for all $N \rightarrow M$ symmetric cloners.* Once again, no limitations were made on the dimensionality of the ancilla. It should be noted that in the case of an ancilla-free system, $X_{B_i^{(a)}} = 1$ for all i, a. Therefore the two expressions are equivalent for *all* ancilla-free cloners.

2-level ancilla

Let us consider as an example that of a UQC with a 2-level ancilla, specifically that of the optimum UQC proposed by Buzek and Hillery [10]. Once again the signal is taken to be $|0\rangle$:

$$\begin{aligned} |\varphi\rangle_{output} &= \sqrt{\frac{2}{3}}|00\rangle \otimes |\uparrow\rangle_A \\ &+ \sqrt{\frac{1}{6}}(|01\rangle + |10\rangle) \otimes |\downarrow\rangle_A \end{aligned} \tag{17}$$

where the subscript A specifies the ancilla. The simple expression for fidelity gives us:

$$F_s = \frac{2 \times \frac{2}{3} + 1 \times \frac{2}{6}}{2 \times \frac{2}{3} + 2 \times \frac{2}{6}} = \frac{5}{6} \tag{18}$$

The reduced density matrix of the clone is

$$\varrho_{red.} = \frac{5}{6}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| \tag{19}$$

It is a simple matter to check that the F_T gives the same result as F_S.

Now let us consider the Simon *et al* proposal of cloning via stimulated emission [1]. In this setup the signal beam is aligned on top of one of the output arms of a pumped type-II downconverting crystal whilst the downconverted photon in the other arm serves as the ancilla. Thus we must also take into consideration the characteristic that both the clones of the QC are in the same mode (spatially indistinguishable). Therefore

$$|00\rangle \rightarrow \sqrt{2}|2_0; 0_1\rangle \tag{20}$$

$$|01\rangle + |10\rangle \rightarrow 2|1_0; 1_1\rangle \tag{21}$$

$$|\uparrow\rangle_A \rightarrow |1\rangle_A \tag{22}$$

$$|\downarrow\rangle_A \rightarrow |0\rangle_A \tag{23}$$

where $|2_0;0_1\rangle$ represents two indistinguishable particles with polarization 0, and no particles with orthogonal polarization 1. Hence on normalization we get

$$|\varphi\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}|2_0;0_1\rangle\otimes|1\rangle_A + |1_0;1_1\rangle\otimes|0\rangle_A) \quad (24)$$

We can see here explicitly that there is no way of calculating the reduced density matrix of an output clone in this formulation. On the other hand F_S gives us

$$F_S = \frac{2\times\frac{2}{3}+1\times\frac{1}{3}}{2\times\frac{2}{3}+2\times\frac{1}{3}} = \frac{5}{6} \quad (25)$$

which is what was purported by Simon *et al* [1] and experimentally verified by Lamas-Linares *et al* [2].

This is a compelling argument for the Simon *et al* proposal to qualify as a $1 \to 2$ UQC, with a valid expression for fidelity of cloning. At this juncture, however, the distinction (or equivalence) between cloning and state discrimination needs to be addressed more clearly for this statement to be absolute. Operationally, in order for a device to qualify as a cloner, it is not unreasonable to demand that both output clones be in different modes to allow for manipulation of one clone independently of the other (modulo entanglement). This can be implemented in the Simon *et al* case by placing a 50/50 beam splitter in the output mode, and explicitly selecting different port coincidences. In theory, this could be achieved using a 'QND filter', where QND number measurements on both ouput ports of the BS could non-destructively select different port ouputs. However, half of all input signals are lost in this process. This gives us:

$$\begin{aligned} |2_0;0_1\rangle &\to \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}(|2_0;0_1\rangle_C + |2_0;0_1\rangle_D) \\ |1_0;1_1\rangle &\to \frac{1}{2}(|01\rangle+|10\rangle) + \frac{1}{2}(|1_0;1_1\rangle_C + |1_0;1_1\rangle_D) \end{aligned} \quad (26)$$

where $|1_0;1_1\rangle_C$ represents one horizontal and one vertical photon in output port C of the BS etc. Plugging this into Eq. (24), discarding same port outputs, and normalizing we get:

$$\begin{aligned} |\varphi\rangle &= \sqrt{\frac{2}{3}}|00\rangle\otimes|1\rangle_A \\ &+ \sqrt{\frac{1}{6}}(|01\rangle+|10\rangle)\otimes|0\rangle_A \end{aligned} \quad (27)$$

i.e. we get exactly the optimum $1 \to 2$ UQC as given in Eq. (17).

At this point we would like to address the use of the ancilla as an anticlone of the signal, since it seems to give additional information about the input state. Firstly it would be useful to keep in mind that the reduced density matrix of the ancilla in Eq. (27) does not correspond to that of the optimum universal anticlone. The reduced density matrix for the optimum universal anticlone is given by

$$\begin{aligned} \varrho_{red.}|^{inp.=|0\rangle}_{Uanticlone} &= \varrho_{red.}|^{inp.=|1\rangle}_{Uclone} \\ &= \{\varrho_{red.}|^{inp.=|0\rangle}_{Uclone} :: (0 \leftrightarrow 1)\} \\ &= \frac{5}{6}|1\rangle\langle 1| + \frac{1}{6}|0\rangle\langle 0| \end{aligned} \quad (28)$$

where $(0 \leftrightarrow 1)$ implies interchanging the computational states 0 and 1. The reduced density matrix of the ancilla is

$$\varrho_{red.}|^{ancilla} = \frac{2}{3}|1\rangle\langle 1| + \frac{1}{3}|0\rangle\langle 0| \quad (29)$$

which is different from that of the optimum anticlone in Eq. (28). Using Eq. (29), the F_T of the ancilla as an anticlone is 2/3. Taking the average of this with the fidelity of the two clones gives a total fidelity of 7/9. Interestingly, the upper bound for the fidelity of a $1 \to 3$ UQC [12] is also 7/9. Thus Eq. (27) could also represent an isotropic asymmetric $1 \to 3$ ancilla-free cloner (or more precisely, if Type I downcoversion were utilized).

We can also see that attempting to include the ancilla in F_S in the Simon *et al* scheme of Eq. (24) leads to the same result:

$$\begin{aligned} F_S &= \frac{Clones + Anticlones}{Clones + Anticlones + Noise} \\ &= \frac{3\times\frac{2}{3}+1\times\frac{1}{3}}{3\times\frac{2}{3}+3\times\frac{1}{3}} = \frac{7}{9} \end{aligned} \quad (30)$$

This may not be so surprising, since as shown above, the two expressions of fidelity are analogous for *all* ancilla-free cloners. However, it must be kept in mind that we have used an average over individual fidelities in calculating F_T for asymmetric cloners.

Under what conditions can the ancilla be considered to be the optimum universal anticlone of the incoming state? The reduced density matrix for the ancilla is

$$\varrho_{red.}|^{ancilla} = |\alpha|^2|A\rangle\langle A| + |\gamma|^2|C\rangle\langle C| + 2|\beta|^2|B\rangle\langle B| \quad (31)$$

In order for this to equate to the optimum anticlone of Eq. (8) we must have $\beta = 0$ and $|A\rangle = |1\rangle$, $|C\rangle = |0\rangle$. We can therefore see that it is almost equivalent to creating a $1 \to 3$ ancilla-free universal quantum cloner, having a shrinking factor of zero as expected.

Conclusion

In this paper a simple expression for the experimental measurement of fidelity has been proposed and for the case of pure input states was shown to be equivalent to the traditional theoretical expression for fidelity for *all* symmetric N$\to$M QCs, with the symmetry requirement being lifted for ancilla-free N$\to$M QCs. The fidelity was explicitly calculated for the case of the optimal UQC [10] and for the case of QC via stimulated emission [1, 2]. An attempt was made to provide greater insight into the physical interpretation of Fidelity and to reconcile somewhat experiments with the theory. It was suggested that although the Simon *et al* proposal appears to give the fidelity for an optimal UQC, it seems to be more closely associated with state discrimination since both outputs are in the same mode.

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