Physics 350 **Spring (12) 2002**

Gaussian Integrals

The function e^{-ax^2}, known as the Gaussian function, shows up often in both physics and probability theory (it's the "normal" probability distribution). In particular, one often encounters integrals of the following form:

$$\int_{-\infty}^{\infty} x^n e^{-ax^2}\,dx = \begin{cases} 0 & \text{n odd} \\ 2I_n(a) & \text{n even} \end{cases}$$

$$\text{where } I_n(a) \equiv \int_0^{\infty} x^n e^{-ax^2}\,dx$$

Values for these "Gaussian" integrals (for both even and odd n) are given as follows:

$$I_0(a) = \frac{1}{2}\sqrt{\frac{p}{a}} \qquad I_1(a) = \frac{1}{2a}$$

$$I_2(a) = \frac{1}{4}\sqrt{\frac{p}{a^3}} \qquad I_3(a) = \frac{1}{2a^2}$$

$$I_4(a) = \frac{3}{8}\sqrt{\frac{p}{a^5}} \qquad I_5(a) = \frac{1}{a^3}$$

No need to continue with this table since, given $I_0(a)$ and $I_1(a)$, one can readily determine all other $I_n(a)$ results using the relation: $I_{n+2}(a) = -\partial I_n(a)/\partial a$. (This relation is easily verified by examining the above integral definition of $I_n(a)$ and noting that the derivative with respect to α can be moved inside the integral.)

Evaluation of $I_1(a)$: This is an elementary integral, most easily done by making the change of variable $z = x^2$, and thus $dx = dz/2x$, which results in

$$I_1(a) = \int_0^{\infty} x e^{-ax^2}\,dx = \frac{1}{2}\int_0^{\infty} e^{-az}\,dz = \left.\frac{-e^{-az}}{2a}\right|_0^{\infty} = \frac{1}{2a}$$

Evaluation of $I_0(a)$: This integral can be done using a trick due to Laplace. For this we consider the square of the function $I_0(a)$, which can be written as

$$[I_0(a)]^2 = \left[\int_0^{\infty} e^{-ax^2}\,dx\right]\left[\int_0^{\infty} e^{-ay^2}\,dy\right] = \int_0^{\infty}\int_0^{\infty} e^{-a(x^2+y^2)}\,dxdy$$

The above two dimensional integral can be evaluated in polar coordinates. Letting $r^2 = x^2 + y^2$ and noting that $dxdy = rdrdq$ and that the integral only covers the first quadrant, we have

$$[I_0(a)]^2 = \int_0^{p/2} dq \int_0^{\infty} e^{-ar^2}\,rdr = \frac{p}{2} I_1(a) = \frac{p}{4a}$$

and thus $I_0(a) = \frac{1}{2}\sqrt{\frac{p}{a}}$ as advertised.