Problem 1

Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

for $a > b > 0$.

Problem 2

Find a solution in power series of the differential equation

$$z\frac{d^2u}{dz^2}+(c-z)\frac{du}{dz}-au=0$$

which is analytic at $z=0$. What is the behavior of the irregular solution at the origin?

Problem 3

Let A be a matrix of order n satisfying the equation

$$A^{k+1} = A^k + A^{k-1}$$

a) Assuming that they are real, find the possible eigenvalues of A.

b) Let A be such a matrix of order 12. Its trace is given by $TrA = 3 + 2\sqrt{5}$. Find the degeneracy of each of its eigenvalues.

c) Find an explicit 2×2 symmetric but non-diagonal matrix obeying the above equation.

Problem 4

Let x and y be two discrete random variables Poisson distributed with parameters λ_1 and λ_2 respectively.

a) Derive the distribution of $z = x + y$.

b) Find the distribution of x given the value of z.

Problem 5

Dust particles in the solar system are "blown" out by solar radiation pressure.

a) Calculate the radius of a spherical particle of uniform density which is just barely blown out. Let M be the mass of the sun and L its total luminosity.

b) Assume a reasonable density for the dust grain and give an order of magnitude estimate for the radius calculated above. Note: $M \approx 2\times10^{33}$g, $L \approx 4\times10^{33}$ erg/s, and Newton's gravitational constant $G = 6.7\times10^{-8}$ dyne cm^2 / g^2.

c) A dust particle twice the radius of the "blown out" particle in parts a) and b) will orbit the sun. Because it is moving through the radially-streaming solar photons, it will experience a drag force. Estimate the time it will take for the particle to spiral into the sun if the initial orbit is circular with a radius of $1AU = 1.5\times10^{13}$cm. (This drag force was first calculated by Poynting and Robertson.)

Problem 6

Consider a parallel plate capacitor filled with a linear dielectric whose permittivity varies as $\varepsilon_1 + ax$, where x is the distance from one plate and where a is chosen so that the permittivity is ε_1 at one plate and ε_2 at the other. The area of the plate is A and the spacing is b. Assume that the size of the plate is large compared to the spacing and ignore fringe fields.

a) What is the capacitance of the system?

b) Find the induced polarization P everywhere in the dielectric.

c) Determine the bound charge density everywhere in the dielectric (i.e., both the volume charge density in the bulk and the surface charge density at the plates).

Problem 7

Consider a very long straight cylindrical wire of length L and radius a carrying a constant steady current I uniformly distributed over the cross sectional area of the wire. If the wire has a uniform resistance per unit length, R/L, then there will be a voltage drop down the length of the wire, $V=IR$, and an electric field in the wire $E=V/L$.

Compute the Poynting vector on the surface of the wire. Find the rate of electromagnetic energy flowing through the surface of the wire. (Assume that L is so long that you may ignore effects at the ends of the wire.) Does energy flow into or out of the wire? Your answer should look familiar. Give a physical explanation for your result.

Problem 8

An infinitely thick metal has a plane surface at z=0 in the xy plane. The metal is characterized by a frequency-dependent conductivity $\sigma(\omega)$. Assume that the response from the conduction electrons dominates that of the bound electrons so that the effective dielectric function can be taken as

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega}$$

and that the permeability $\mu = 1$.

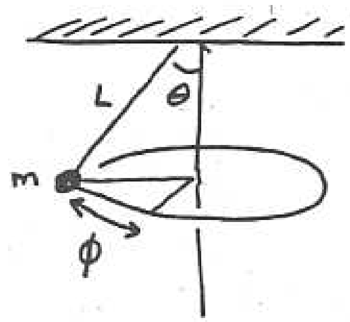
a) An electromagnetic plane wave of angular frequency ω is incident normally on the metallic surface at z=0. Find the reflection coefficient $r(\omega) = E_r / E_i$, where E_i is the amplitude of the incident wave and E_r that of the reflected wave. Assume ω to be such that $\frac{4\pi\sigma(\omega)}{\omega} >> 1$.

b) Compute the "skin depth" for the wave penetrating into the metal.

Problem 9

A pendulum consists of a point mass m at the end of a massless string of length L. The motion of the pendulum is not confined to a plane.

a) Write the Lagrangian for the system using the generalized coordinates θ and ϕ shown in the figure.



b) The pendulum is set in motion such that the mass m executes a circular orbit at an angular velocity $d\phi/dt = \omega$. Find an equation giving the corresponding value of the angle θ.

c) The pendulum is perturbed from its circular orbit by a very small impulse in the θ direction. Find the frequency of the resulting small amplitude oscillations.

Problem 10

Consider the process

$$K \to \pi^0 + \pi^0 + \pi^0.$$

Let the momenta of the kaon and the pi mesons be p, p_1, p_2, p_3 respectively. Define the Lorentz invariant quantities $x = (p_1 + p_2)^2$ and $y = (p_2 + p_3)^2$. There is a region in the xy plane (for fixed p) within which all the events must lie. Find the equation for the boundary of this region in terms of the masses m_K and m_π of the kaon and pion. This representation of events is called the 'Dalitz plot'.