Lecture #1

Classical Mechanics – A.N. Jordan

We start with the Schrödinger Equation:

$$i\partial_t |\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \quad w/ \quad \hat{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V(\hat{x})$$

$$= +\frac{\hat{p}^2}{2m} + V(\hat{x})$$

We can formally integrate this equation to find the propagator:

$$|\psi(t)\rangle = \hat{\mathcal{U}}|\psi(0)\rangle, \quad \hat{\mathcal{U}} = e^{-i\hat{H}t/\hbar}$$

This is nice and simple, and indeed if there are energy eigenstates $\{E_n\}, |\phi_n\rangle$, then

$$|\psi(t)\rangle = \hat{\mathcal{U}}\sum_n |\phi_n\rangle\langle\phi_n|\psi(0)\rangle$$

$$= \sum_n e^{-iE_n t/\hbar} C_n|\phi_n\rangle, \quad C_n = \langle\phi_n|\psi(0)\rangle$$

as usual. However, let us stay in the position basis, then,

$$\psi(t,x) = \langle x|\psi(t)\rangle = \langle x|\hat{\mathcal{U}}|\psi(0)\rangle, \text{ but}$$

$$e^{[\hat{p}^2/2m + V(\hat{x})]t} \neq e^{\frac{\hat{p}^2}{2m}t} e^{V(\hat{x})t}, \text{ so no}$$

simplification!

$$= \int dx' \langle x|\hat{\mathcal{U}}^t|x'\rangle \psi(x')$$

Trick! first, warm up with $V=0$, so

$$\hat{\mathcal{U}}(t) = e^{-i\hat{p}^2 t/2m\hbar}$$

, now $\hat{p}^2$ commutes with itself, so

$$= e^{-i\hat{p}^2 \frac{N\varepsilon}{2m\hbar}} \text{, when } N\varepsilon = t,$$

$$= e^{-i\hat{p}^2 \sum_{i=1}^{N} \varepsilon/2m\hbar} = \prod_{i=1}^{N} e^{-i\hat{p}^2 \frac{\varepsilon}{2m\hbar}}$$

at time $t=\varepsilon$, $t=2\varepsilon$, $t=3\varepsilon$, etc. we introduce a complete set of position states $1 = \int dx_i\, |x_i\rangle\langle x_i|$

$$\therefore \psi(x,t) = \langle x|\hat{\mathcal{U}}|\psi(0)\rangle = \int dx'\, \langle x|\hat{\mathcal{U}}|x'\rangle \psi(x',0)$$

$\langle x|\mathcal{U}^t|x'\rangle$ is the position-space propagator.

use my trick

$$\langle x|\mathcal{U}^t|x'\rangle = \prod_{i=1}^{N} \int dx_i\, \langle x|x_i\rangle\langle x_i| e^{-i\hat{p}^2\frac{\varepsilon}{2m\hbar}} |x_{i+1}\rangle\langle x_{i+1}|x'\rangle$$

~~Because we can take ε as small as we like~~

$$\cancel{\lim_{\varepsilon\to 0}\langle x_i| e^{-i\hat{p}^2\frac{\varepsilon}{2m\hbar}} |x_{i+1}\rangle \simeq \langle x_i|\left[1 - \frac{i\varepsilon}{2m\hbar}\hat{p}^2 + O(\varepsilon^2)\right]|x_{i+1}\rangle}$$

$$\cancel{\langle x_i|\hat{p}^2|x_{i+1}\rangle = -\hbar^2\langle x_i|\partial_x^2|x_{i+1}\rangle}$$

Insert complete set of ~~position~~ momentum states!

$$\langle x_i | e^{-i\frac{\hat{p}^2 \varepsilon}{2m\hbar}} | x_{i+1}\rangle$$

$$= \int \frac{dp_i}{2\pi\hbar} \langle x_i|p_i\rangle e^{-i\frac{p_i^2\varepsilon}{2m\hbar}} \langle p_i | x_{i+1}\rangle$$

$\langle x|p\rangle = e^{ipx/\hbar}$

$$= \int \frac{dp_i}{2\pi\hbar}\, e^{i x_i \frac{p_i}{\hbar} - i \frac{p_i^2 \varepsilon}{2m\hbar} - i p_i \frac{x_{i+1}}{\hbar}}$$

$$= \int_{-\infty}^{\infty} \frac{dp_i}{2\pi\hbar}\, e^{-i\frac{p_i^2\varepsilon}{2m\hbar} + i \frac{p_i}{\hbar}(x_i - x_{i+1})}$$

But this is a Gaussian integral!

$$= \sqrt{\frac{m}{2\pi i\varepsilon\hbar^2}}\, \exp\left[\frac{im}{2\varepsilon\hbar}(x_i - x_{i+1})^2\right]$$

$$\left[\int e^{-ax^2+bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}\right]$$

$$\therefore \langle x_N | U^t | x_0\rangle = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i\, \exp\left[i \sum_{l=1}^{N} \frac{m(x_i - x_{i+1})^2}{2\varepsilon}\right]$$

Now, if we wish, we can evaluate these integrals 1 at a time, i.e.,

$$\int_{-\infty}^{\infty} dx_1\, \exp\left[\frac{im}{2\varepsilon}(x_1-x_0)^2 + (x_2-x_1)^2\right]$$

$$= \sqrt{\frac{2\pi i\varepsilon}{2m}}\, \exp\left[\frac{im}{2\cdot 2\varepsilon}(x_2-x_0)^2\right]$$

$$\int_{-\infty}^{\infty} dx_2\, \exp\left[\frac{im}{2\varepsilon}\left(\frac{1}{2}(x_2-x_0)^2 + (x_3-x_2)^2\right)\right] = \sqrt{\frac{2\pi i\varepsilon\cdot 2}{3m}}\, e^{\frac{im}{3\cdot 2\varepsilon}(x_3-x_0)^2}$$

And so, on, Gaussian all the way, until

$$\langle x | \mathcal{U}^t | x' \rangle = \left(\frac{2\pi i \varepsilon}{m}\right)^{\frac{N-1}{2}} \left(\frac{2\pi i \varepsilon}{m}\right)^{-\frac{N}{2}} \frac{1}{\sqrt{N}} \exp\left[\frac{im}{2N\varepsilon}(x'-x)^2\right]$$

$$= \sqrt{\frac{m}{2\pi i \varepsilon N}} \exp\left[\frac{im}{2N\varepsilon \hbar}(x'-x)^2\right]$$

but $N\varepsilon = t$, so we recover the free propagator.

Why did I do this?

Because it is now easy to go back and put in $V(\hat{x})$:

$$\langle x_i | e^{-\frac{i\varepsilon \hat{H}}{\hbar}} | x_{i+1} \rangle \simeq e^{-\frac{i\varepsilon V(x_i)}{\hbar}} \langle x_i | e^{-\frac{i\varepsilon \hat{p}^2}{2m}} | x_{i+1} \rangle$$

$$+ \mathcal{O}(\varepsilon^2)$$

$$\therefore \langle x | \mathcal{U}^t | x' \rangle = \lim_{\varepsilon \to 0} \left(\frac{m}{2\pi i \varepsilon}\right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dx_{N-1} \cdots dx_1$$

$$\times \exp\left[\frac{i}{\hbar}\left(\sum_{l=1}^{N} m\frac{(x_l - x_{l-1})^2}{2\varepsilon} - \varepsilon V(x_l)\right)\right]$$

In the limit when $\varepsilon \to 0$, $N \to \infty$

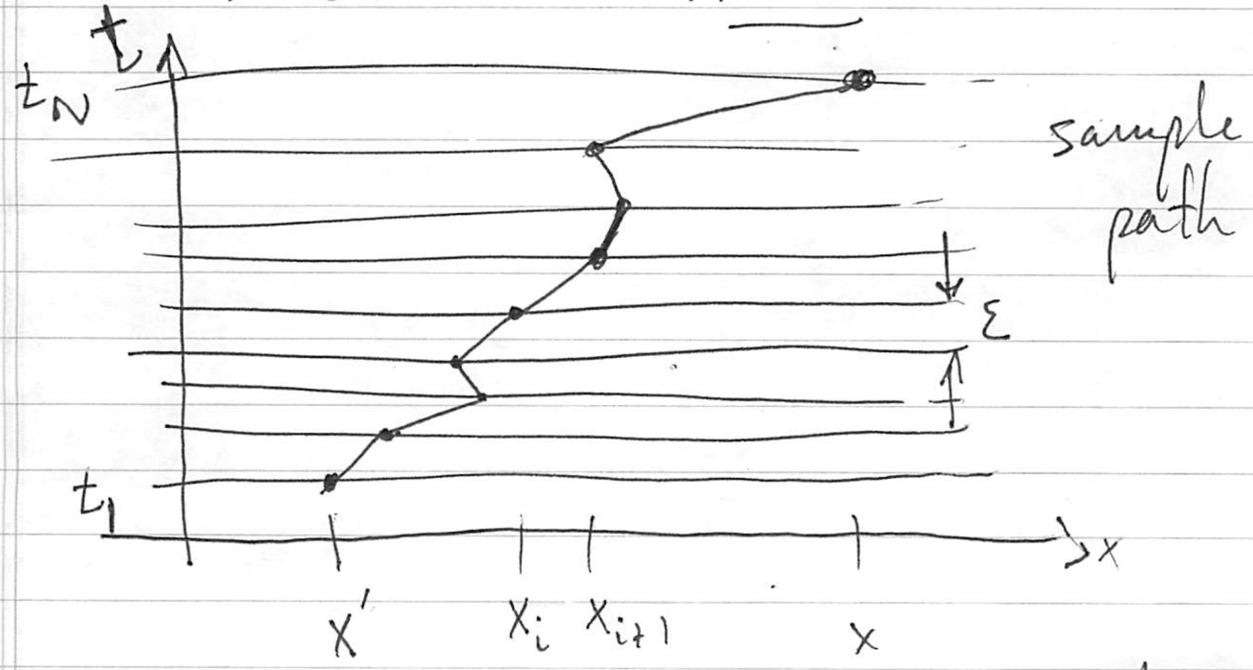
$$\frac{x_\ell - x_{\ell-1}}{\varepsilon} \to \dot{x}_\ell \,, \text{ so } \frac{(x_\ell - x_{\ell-1})^2}{\varepsilon} \to \varepsilon (\dot{x}_\ell)^2$$

thus, we define $\left(\frac{m}{2\pi i \varepsilon}\right)^{N/2} \int_{-\infty}^{\infty} dx_{N-1} \cdots dx_1 \equiv \int \mathcal{D}[x]$,

so $\langle x | \mathcal{U}^t | x' \rangle = \int \mathcal{D}[x] \exp\left[\frac{i}{\hbar} S[x(t)]\right]$,

where $S[x(t)] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \dot{x}^2(t) - V(x(t))\right]$

is called the Action



propagator calls for a sum over all paths in space, weighted by action of that path.

Now, we come to classical physics, taken as the limit $\hbar \to 0$.

In this limit, we can evaluate each integral in the stationary phase Approximation because slight path changes cause a huge phase oscillation which tends to cancel out $e^{iS/\hbar}$, $\hbar \to 0$.

There will be little or no contribution to the path *summation* **Except** for trajectories $\bar{X}(t)$ for which the action is stationary. This means that if I vary the path δx, the action doesn't change:

$$\left.\frac{\delta S[x(t)]}{\delta x}\right|_{x(t)=\bar{x}(t)} = 0 \qquad (\star)$$

This condition *defines* the classical trajectories

Example for the ~~free particle~~. Harmonic oscillator

Quantum mechanically we can evaluate each integral in the stationary phase approximation,

$$S = S_{cl} + \cancel{\left.\frac{\delta S}{\delta x}\right|_{x_{cl}}}^{\,0} (x - x_{cl}) + \frac{1}{2}\frac{\delta^2 S}{\delta x^2}(x - x_{cl})^2 + \cdots$$

and work in the semi-classical Approx.

For our purposes, $\hbar = 0$, so all we need are the criteria to find the unique trajectory that satisfies Eq. (*).

Foundation of Mechanics

We define $L = \underbrace{\frac{1}{2} m \dot{x}^2}_{K\text{, kinetic energy}} - V(x)$ as the Lagrangian of the mechanical system, and

$$S = \int_{t_1}^{t_2} dt\, L$$

as the action.

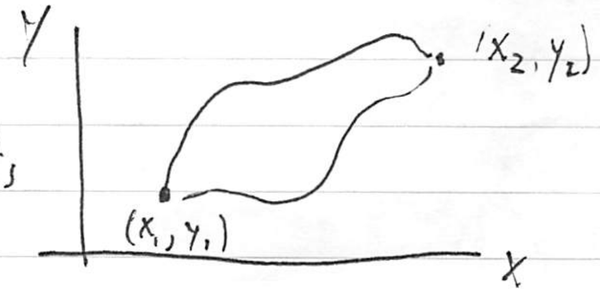
Hamilton's principle is that the Dynamical trajectory is defined by $\delta S = 0$.

Refresher on Variational Calculus.

Let's simplify to one-Dimension.

suppose we have a function $f(y, y', x)$, defined on a path $y = y(x)$ between two values x_1 and x_2, and $y' = \frac{dy}{dx}$.

We want to find a path $y(x)$, so the line integral $J = \int_{x_1}^{x_2} f(y, y', x)\, dx$ has a stationary value.
+ we keep the end-points fixed.



($x \Leftrightarrow t$ in mechanics)

Let all such paths be labeled by a parameter, α. Let the correct path be $\alpha = 0$.

$$y(x, \alpha) = y(x, 0) + \eta(x) \cdot \alpha$$

where η vanishes at the end points.

we assume all is smooth.

Now, we have $J(\alpha)$, and we want to find the stationary point,

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0.$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} dx \left\{ \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right\}$$

but (2nd term) $\displaystyle\int_{x_1}^{x_2} dx \, \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha} = \int_{x_1}^{x_2} dx \left[-\frac{\partial y}{\partial \alpha}\frac{\partial^2 f}{\partial y' \partial x} \right] + \left.\frac{\partial y}{\partial \alpha}\frac{\partial f}{\partial y'}\right|_{x_1}^{x_2}$

However, $\displaystyle\left.\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} = \eta(x)\Big|_{x_1}^{x_2} = 0$, because η is 0 at end points.

$\therefore$ surface term $= 0$.

$$\therefore \frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) dx \cdot \left(\frac{\partial y}{\partial \alpha} \right)$$

Now, we set $\alpha = 0$ to find $y(x,t)$ that makes J stationary.

However, $\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0}$ is arbitrary (Except for end points + continuity)

$$\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0} = \eta(x)$$

Therefore, if $\int_{x_1}^{x_2} M(x)\,\eta(x)\,dx = 0$

where $\eta(x)$ is arbitrary, then $M = 0$ identically.

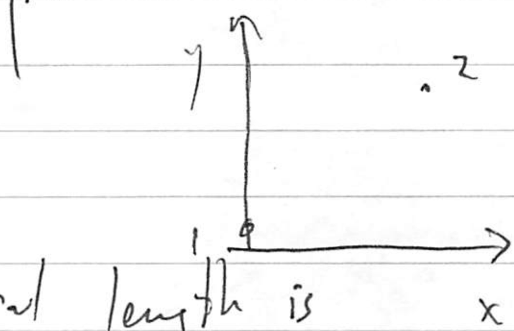
(obvious)

$\therefore$ we have $\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$ as the Eqn. of motion.

We define $\delta y = \left.\frac{\partial y}{\partial \alpha}\right|_{0} d\alpha$

represents the departure from the true path, $\delta J = \left.\frac{\partial J}{\partial \alpha}\right|_{0} d\alpha$, etc.

$\boxed{\text{Example}}$ Find the shortest distance between two points in a plane.



Arclength is

$$ds = \sqrt{dx^2 + dy^2}\ , \text{ Total length is}$$

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\ , \text{ so}$$

so $f = \sqrt{1+(y')^2}$, our Equ. of motion is

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right),\ \cancel{\text{so}}\ 0 = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{1+(y')^2}} \cdot y'\ , \text{ or}$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0 \Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = c\ , \text{ const.}$$

$(y' = \text{const} \equiv a)$

$$(y')^2 = c^2\left(1+(y')^2\right);\ \text{or } \boxed{y' = \frac{c}{\sqrt{1-c^2}} \equiv a}$$

$\boxed{\text{So } y(x) = ax + b}$ straight line.

Home work problem:

Brachistochrone problem

Q: find the curve joining pts 1 & 2, along which a particle falling from rest @ 1 under gravity, reaches pt. 2 in the <u>least</u> amount of <u>time</u>.

Hint: Arclength in example → time here.