Lecture 10

Symplectic Approach - continued:

Recall $\vec{x} = (q_1, q_2, \cdots; p_1, p_2, \cdots, p_n)^T$

$$\dot{\vec{x}} = \mathbb{J} \cdot \frac{\partial H}{\partial \vec{x}}, \quad \text{or} \quad \dot{x}_i = \mathbb{J}_{ij} \frac{\partial H}{\partial x_j}$$

[time-independent] Define a ~~conomical~~ canonical transformation

$\vec{\zeta}(\vec{x})$, now $\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial x_j} \dot{x}_j$

or $\dot{\vec{\zeta}} = \mathbb{M} \dot{\vec{x}}$, $M_{ij} = \frac{\partial \zeta_i}{\partial x_j}$ is the Jacobian of the transformation.

therefore $\dot{\vec{\zeta}} = \mathbb{M} \cdot \mathbb{J} \cdot \frac{\partial H}{\partial \vec{x}}$

I can express $\frac{\partial H}{\partial x_i} = \frac{\partial H}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial x_i} = \frac{\partial H}{\partial \zeta_j} M_{ji}$

or $\frac{\partial H}{\partial \vec{x}} = \mathbb{M}^T \cdot \frac{\partial H}{\partial \vec{\zeta}}$

Thus $\dot{\vec{\zeta}} = \mathbb{M} \cdot \mathbb{J} \cdot \mathbb{M}^T \cdot \frac{\partial H}{\partial \vec{\zeta}}$

for a time-independent transforms $K(\vec{\zeta}) = H(\vec{\zeta}(\vec{x}))$

Thus $\dot{\vec{\zeta}} = \mathbb{J} \cdot \frac{\partial K}{\partial \vec{\zeta}}$

So, the transformation is canonical if

$$\boxed{\mathbb{M} \cdot \mathbb{J} \cdot \mathbb{M}^T = \mathbb{J}}$$

or $\mathbb{M} \cdot \mathbb{J} = \mathbb{J} (\mathbb{M}^T)^{-1}$; multiply by $\mathbb{J}$ from L & $-\mathbb{J}$ from R

$\mathbb{J} \cdot \mathbb{M} = (\mathbb{M}^{-1})^T \cdot \mathbb{J}$

or $$\boxed{\mathbb{M}^T \cdot \mathbb{J} \cdot \mathbb{M} = \mathbb{J}}$$ because $\mathbb{J}^2 = -\mathbb{1}$

Poisson Brackets:

Define $\{u, v\} = \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}$

$= \left(\frac{\partial u}{\partial \vec{x}}\right)^T \cdot \mathbb{J} \cdot \frac{\partial v}{\partial \vec{x}}$; Notice

$\{q_j, q_k\} = 0$

$\{p_j, p_k\} = 0$

$\{q_j, p_k\} = \delta_{jk}$

or $\{\vec{x}, \vec{x}\} = \mathbb{J}$

However, $\{\vec{\zeta}, \vec{\zeta}\} = M^T \cdot \mathbb{J} \cdot M = \mathbb{J}$

Same as $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$

$\Rightarrow$ "fundamental" Poisson Brackets are invariant under canonical transformations.

(arbitrary functions)

Consider $u, v(\vec{x})$, then

$$\frac{\partial v}{\partial \vec{x}} = \mathbb{M}^T \cdot \frac{\partial v}{\partial \vec{z}}, \quad \frac{\partial u}{\partial \vec{x}}^T = \left(\mathbb{M}^T \frac{\partial u}{\partial \vec{z}}\right)^T$$

$$= \left(\frac{\partial u}{\partial \vec{z}}\right)^T \cdot \mathbb{M}$$

So $$\{u, v\} = \left(\frac{\partial u}{\partial \vec{x}}\right)^T \cdot \mathbb{J} \cdot \frac{\partial v}{\partial \vec{x}}$$

$$= \left(\frac{\partial u}{\partial \vec{z}}\right)^T \cdot \underbrace{\mathbb{M} \cdot \mathbb{J} \cdot \mathbb{M}^T}_{\mathbb{J}} \frac{\partial v}{\partial \vec{z}}$$

$$= \left(\frac{\partial u}{\partial \vec{z}}\right)^T \cdot \mathbb{J} \cdot \frac{\partial v}{\partial \vec{z}}$$

$\therefore \{u, v\}_{\vec{x}} = \{u, v\}_{\vec{z}} \Rightarrow$ canonical Invariant.

(No need to specify subscript)

properties: ① $\{u, u\} = 0$, ② $\{u, v\} = -\{v, u\}$

③ $\{au + bv, w\} = a\{u, w\} + b\{v, w\}$

④ $\{uv, w\} = \{u, w\}v + u\{v, w\}$

⑤ $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$ Jacobi identity.

Equations of Motion + Conservation Thms. w/ the Poisson Bracket.

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial u}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

<u>or</u> $$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}$$

when $u = X_i \Rightarrow \dot{X}_i = \{X_i, H\} = \mathbb{J} \cdot \frac{\partial H}{\partial x}$

$\Rightarrow$ recover Hamilton's Eqns.

Next, let $u = H$

$$\frac{dH}{dt} = \cancel{\{H, H\}}^{0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

If Q_i, P_i are valid canonical transformations, then

$$\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0$$

$$\{Q_i, P_j\} = \delta_{ij}$$

★ [important]

A basis property of Hamilton's Equation is that they preserve the $2N$-dimensional volumes in phase space:

Recall $\dot{\vec{X}}_i = \mathbb{J}_{ij} \frac{\partial H}{\partial x_j}$, $\mathbb{J} = \left(\begin{array}{c|c} 0_N & \mathbb{1}_N \\ \hline -\mathbb{1}_N & 0_N \end{array} \right)$

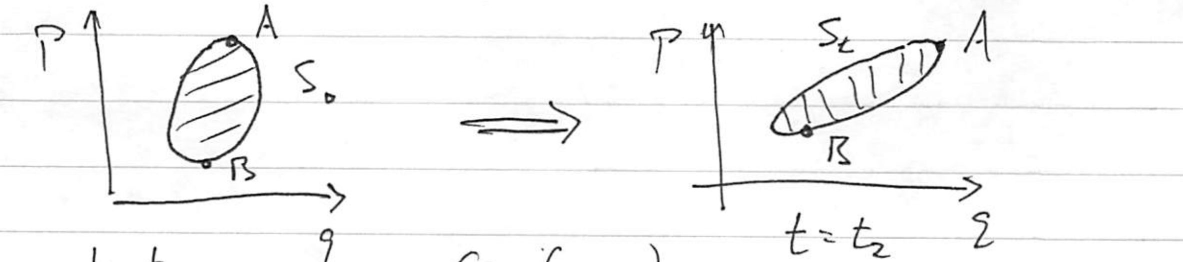
or $\dot{\vec{X}} = \mathbb{J} \cdot \frac{\partial H}{\partial \vec{X}}$; Consider $\vec{\nabla}_x \cdot \dot{\vec{X}}$ in phase space,

$$\vec{\nabla}_x \cdot \dot{\vec{X}} = \sum_i \nabla_i \dot{X}_i = \sum_{i,j} \nabla_i \mathbb{J}_{ij} \nabla_j H$$

$$= \vec{\nabla} \cdot \mathbb{J} \cdot \vec{\nabla} H = \frac{\partial}{\partial \vec{p}} \cdot \left(-\frac{\partial H}{\partial \vec{q}} \right) + \frac{\partial}{\partial \vec{q}} \cdot \left(\frac{\partial H}{\partial \vec{p}} \right) = 0$$

or $\vec{\nabla}_x \cdot \dot{\vec{X}} = 0$ (incompressible phase space)

This leads us to Liouville's Theorem:



(surface)

Consider a closed curve S_0 @ $t = t_1$.
Evolve each point on the curve dynamically in time to get a new curve (surface) S_t.

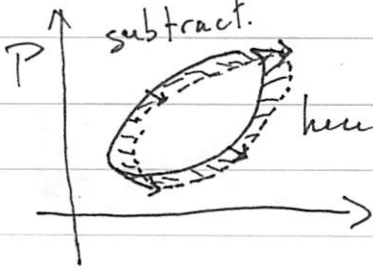
Thm: the Volume in S_t = the Volume in S_0

Proof: let $V_t = \int_{S_t} d^{2N} X = \int_{S_t} dx_1 dx_2 \cdots dp_1 dp_2 \cdots$

Let S_t = closed surface of points @ time t.

~~Then~~ Consider $\frac{d}{dt} V(t) = \frac{d}{dt} \int_{S_t} d^{2N} X$ (volume integral)

$= \oint_{S_t} \frac{d\vec{x}}{dt} \cdot d\vec{S}$, where $\oint_{S_t}$ is a surface integral over the Boundary.



here I add ⇒ ∂ only!

This is because ~~$\int_{S_t} d^{2N} x = \int_{S_t} \nabla_{\vec{x}} \cdot \vec{x}$~~

$$\frac{d}{dt}\int [dV] = \frac{d}{dt}\int_{S_t} [dq_1 \, dq_2 \cdots dq_N \, dp_1 \, dp_2 \cdots dp_N]$$

$= \oint_{S_t} \sum_i \frac{dx_i}{dt} \, d\bar{x}$; $d\bar{x}$ ⇒ all other coordinates ⇒ $\perp$ to dx_i

thus $= \oint_{S_t} \dot{\vec{X}} \cdot d\vec{S}$; However by the $\boxed{\text{divergence Thm.}}$

$\therefore = \int_{S_t} (\vec{\nabla}_{\vec{x}} \cdot \dot{\vec{X}}) \, d^{2N} x$ (Volume integral)

but $\nabla_{\vec{x}} \cdot \dot{\vec{X}} = 0$ from the previous page,

so $\frac{dV(t)}{dt} = 0$ ⇒ phase-space volume is conserved