Physics 418
Homework 6 - Due March 6, 2009

Problem 1: Suppose that a complete description of a physical process is not possible due to some missing information. We can think of this missing information as characterizing our ability to predict an event. Consider, for example, a loaded die. If the die is strongly weighted, then it will always land on 6 (for example), so we have no missing information. However, if the weighting is imperfect, sometimes it will land on another side (say 1) (and we are not given the detailed trajectory that the die took) with small probability, and we will be surprised. Our ability to predict that event is quite low, so our missing information is very high. It is useful for shifty gamblers (and therefore physicists) to define a missing information function I that depends on the probability of an event P. This missing information function may also be thought of as the surprisal value of an event. The missing information function I of seperate events $E_1, E_2, \ldots, E_N$ which are described by the probabilities $\{P_j\}, j = 1, \ldots, N$ should have these properties:

- $I(P = 1) = 0$: if there is only one possible event, there is no missing information.
- $I(P)$ is a smooth function of the probability P.
- $I(PQ) = I(P) + I(Q)$. In other words the missing information of two *independent* events (with probability P and probability Q) occurring is the sum of the missing information of both of those events.

(a) What is the missing information I if an event E of probability P occurs?
(b) What is the average missing information of all possible events $E_1, E_2, \ldots, E_n$? Show this quantity is the Shannon entropy mentioned in class.

Problem 2: Starting with the Shannon entropy, $S = -k \sum_j P_j \ln P_j$, apply this entropy to an arbitrary statistical mechanical system. The constant k is taken to be the Boltzmann constant, and the only information we are given about the system is that the total internal energy is U, and the total particle number is N. Find the probability distribution $\{P_j\}$ that maximizes the average missing information S, subject only to the information that we are given above.

Problem 3: Revisit the Fermi oscillator problem from the previous homework (recall this is a model where there are N sites that contains a two-level quantum system that has energies $E_0 = 0$ and $E_1 = \epsilon$). (a) Evaluate the grand canonical partition function for this system. (b) Find the thermodynamic quantities $P, N, U, S, U/N, S/N$ from your result in part (a) in the thermodynamic limit. How do these results compare with what you found in the canonical ensemble?