The Biphoto Birth Zone
in Spontaneous Parametric Down-Conversion

Foundations and Applications

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Outline

• Important concepts in SPDC
• Theoretical treatment of correlations
• Experimental methods
• EPR-steering with these correlations
• Application: Monogamy of EPR correlations
What is SPDC?

• **Spontaneous:** There’s no seeding or stimulation
• **Parametric:** Optical parameters temporarily altered by pump beam
  – The final quantum state of the medium is left unchanged, though
• **Down-Conversion:** Pump photons are down-converted into correlated signal-idler photon pairs.
  – The energies and frequencies of the signal and idler photons are below that of the pump photon.
What Correlations are there?

Optical Parametric processes Conserve...

Energy: \( \omega_{\text{pump}} = \omega_1 + \omega_2 \quad : \quad E = \hbar \omega \)

Momentum: \( \vec{k}_{\text{pump}} = \vec{k}_1 + \vec{k}_2 \quad : \quad \vec{p} = \hbar \vec{k} \)

If \( \omega_{\text{pump}} \) and \( \vec{k}_{\text{pump}} \) \( \approx \) constant...

\( \Rightarrow \) Energy and momentum correlations!
The Biphotoon Birth Zone

- The region where the signal-idler photons are likely to be found given where the destroyed pump photon was:

\[
\Delta_{BZ} \equiv 2\sigma(x_1|x_1+x_2)
\]

\[
\Delta_{pump} = 2\sigma\left(\frac{x_1+x_2}{2}\right)
\]

- The Birth zone number:

\[
N \equiv \left(\frac{\Delta_{pump}}{\Delta_{BZ}}\right)^d
\]

(a measure of the strength of these correlations)
How is SPDC even possible?

- Standard (linear) optics:

What does light do to matter?
How is SPDC even possible?

• Standard (linear) optics:
  – EM waves slightly disturb bound electrons in a solid
    • They recoil back and forth like springs.
  – Steady state response is proportional to input electric field.
How is SPDC even possible?

- **Standard (linear) optics:**
  - EM waves slightly disturb bound electrons in a solid
    - They recoil back and forth like springs.
  - Steady state response is proportional to input electric field.

- **Nonlinear optics:**
  - Sufficiently intense light pushes and pulls electrons hard/far enough that they can’t be treated so simply.
  - Steady state response is no longer proportional:
    - You can have different frequency components in the output
  - Examples:
    - Optical Rectification
    - Second harmonic generation
In four (easier-said-than-done) steps:

1. Quantize the electromagnetic field

\[ [\hat{A}, \hat{B}] = i\hbar\{A, B\} \]
The Birth Zone Width from Quantum Optics

In four (easier-said-than-done) steps:

1. Quantize the electromagnetic field

2. Find the Hamiltonian for the SPDC process

\[ H_{EM} = \frac{1}{2} \int d^3r \left( \vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H} \right) \]
The Birth Zone Width from Quantum Optics

In four (easier-said-than-done) steps:

1. Quantize the electromagnetic field

2. Find the Hamiltonian for the SPDC process

3. Use the Schrödinger equation to get the biphoto state \( |\Psi_{SPDC}\rangle \).
   - The spatially varying part is the biphoto wavefunction
     \[
     i\hbar \frac{d}{dt} |\psi\rangle = H_{EM} |\psi\rangle
     \]
In four (easier-said-than-done) steps:

1. Quantize the electromagnetic field

2. Find the Hamiltonian for the SPDC process

3. Use the Schrödinger equation to get the biphoton state $|\Psi_{SPDC}\rangle$.
   - The spatially varying part is the biphoton wavefunction

4. Calculate the Birth Zone Width from the biphoton wavefunction.
Assumptions:

- The pump beam is well-collimated and narrowband
- The pump beam is bright enough to be treated classically
  - But not so bright as to damage the material or to include multi-photon effects
The Birth Zone Width from Quantum Optics

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• The pump beam is well-collimated and narrowband
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  – But not so bright as to damage the material or to include multi-photon effects
• The crystal is wider than the beam (and a lot wider than an optical wavelength)
• Optical constants are uniform throughout the crystal.
• The crystal is AR-coated (so we needn't consider multiple internal reflections)
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• The crystal is wider than the beam (and a lot wider than an optical wavelength)
• Optical constants are uniform throughout the crystal.
• The crystal is AR-coated (so we needn't consider multiple internal reflections)
• We use frequency filtering to look at the part of the downconverted spectrum where $\omega_1 \approx \omega_2 \approx \frac{\omega_{pump}}{2}$
The Birth Zone Width from Quantum Optics

The Quantum Biphoto State from SPDC

\[ |\Psi_{\text{SPDC}}\rangle \approx C_0 |0_1, 0_2\rangle + C_1 d_{\text{eff}} \sqrt{I_{\text{pump}} T^2} \int d^3 k_1 d^3 k_2 \Phi(\vec{k}_1, \vec{k}_2) \hat{a}^\dagger(\vec{k}_1) \hat{a}^\dagger(\vec{k}_2) |0_1, 0_2\rangle \]

The Biphoto Wavefunction

\[ \Phi(\vec{k}_1, \vec{k}_2) = N \text{ Sinc} \left( \frac{\Delta k_z L_z}{2} \right) \nu(k_{1x} + k_{2x}, k_{1y} + k_{2y}) \]

\[ :\Delta k_z = k_{1z} + k_{2z} - k_{\text{pump} z} \]

What can we learn about rate \( R \) of produced photon pairs?

- \( R \propto d_{\text{eff}}^2 \) (high nonlinearities are especially important)
- \( R \propto I_{\text{pump}} \) (pump photons in \( \rightarrow \) output pairs out)
- \( R \propto L_z^2 \) (The amplitude is a sum over the paths in the crystal)
The Biphoton Wavefunction (in 1D)

• With a Gaussian Pump Beam profile:

\[ \Phi(k_{1x}, k_{2x}) = N \text{ Sinc} \left( \frac{L_z \lambda_{pump}}{8\pi} (k_{1x} - k_{2x})^2 \right) e^{-\sigma_{pump}^2 (k_{1x} + k_{2x})^2} \]

• We can calculate the Transverse Correlation Width \( \sigma(x_1 - x_2) \)

\[ \sigma(x_1 - x_2) = \sqrt{\frac{9L_z \lambda_{pump}}{10\pi}} = \Delta_{BZ} \]

(agrees with experimental data too!)
The Double-Gaussian Approximation

- The exact biphoton wavefunction is hard to work with...

$$\psi(x_1, x_2) = \mathcal{N} \left[ (x_1 - x_2) \sqrt{\pi} \left( S \left( \frac{x_1 - x_2}{2\sqrt{\pi a}} \right) - C \left( \frac{x_1 - x_2}{2\sqrt{\pi a}} \right) \right) + 2\sqrt{a} \left( \cos \left( \frac{(x_1 - x_2)^2}{8a} \right) + \sin \left( \frac{(x_1 - x_2)^2}{8a} \right) \right) \right] e^{-\frac{(x_1 + x_2)^2}{16\sigma_{pump}^2}}$$

$$S(x) = \int_0^x \sin \left( \frac{\pi}{2} t^2 \right) dt \quad C(x) = \int_0^x \cos \left( \frac{\pi}{2} t^2 \right) dt \quad a \equiv \frac{L_z \lambda_{pump}}{4\pi}$$
The Double-Gaussian Approximation

- The exact biphoton wavefunction is hard to work with.
- Gaussian wavefunctions are well studied and have many useful properties.
  - Useful Fourier-Transform properties
    - They saturate uncertainty relations
  - Statistical properties like mutual information are easy to calculate
  - You can actually find *quantum* entropies even though it’s infinite-dimensional

\[
\psi^{DG}(x_1, x_2) = \frac{1}{\sqrt{2\pi\sigma_{pump}\Delta_{PM}}} e^{-\frac{(x_1+x_2)^2}{16\sigma_{pump}^2}} e^{-\frac{(x_1-x_2)^2}{4\Delta_{PM}^2}}
\]
The Double-Gaussian Approximation

- The center of the biphoton wavefunction is well-approximated with a double-Gaussian by matching peaks:
  - (the low-level oscillating wings are not)
- Experiments may neglect wings due to noise floor.

\[ \psi^{DG}(x_1, x_2) = \frac{1}{\sqrt{2\pi \sigma_{pump} \Delta_{PM}}} e^{-\frac{(x_1 + x_2)^2}{16\sigma_{pump}^2 \Delta_{PM}}} e^{-\frac{(x_1 - x_2)^2}{4\Delta_{PM}^2}} \]

\[ \Delta_{PM} = \sqrt{\frac{4 L_z \lambda_{pump}}{9\pi}} \]
(fits Full Width at 48.2% of Max)
How strong are these Correlations?

\[
\sigma(x_1-x_2) = \sqrt{\frac{9L_z \lambda_{pump}}{10\pi}} = \Delta_{BZ} \geq \sigma(x_1|x_2)
\]

- Example:
  - BiBO crystal for Type-I SPDC
  - \(\lambda_{pump}=775\text{nm}\)
  - \(L_z=3\text{mm}\)
  - \(\sigma_{pump}=0.5\text{mm}\)
  \(\Rightarrow \Delta_{BZ}=25.8\mu\text{m}\)

  \(\Rightarrow\) (Birth Zone Number) \(N = (38.75)^2 \approx 1502\)

  \(\Rightarrow\) (Mutual Information) \(h(\vec{x}_1: \vec{x}_2) \approx 9.55\) bits

  \(\Rightarrow\) (Entanglement of Formation): \(\mathcal{E}(\hat{\rho}) \approx 9.4\) ebits

  \(\Rightarrow\) (Pearson R-value) \(R \approx 0.9987\)
How to measure these correlations?

• To measure just $\Delta_{BZ}$ ...
  – Consider the following, based on Howell et al: PRL, 92 210403 (2004).
    • 40$\mu$m slits
    • 390nm pump laser
    • 2mm BBO crystal
  – Our estimate:
    $\Delta_{BZ} \approx 14.9\mu$m
  – Their result (via deconvolution):
    $\Delta_{BZ} \approx 13.5 \pm 2.6\mu$m
How to measure \textit{all} the correlations?

**Hard way:** Measure the coincidences coming from every pair of pixels in transverse position and momentum planes.

Brute force:

\[ M \sim n^4 \quad T \sim n^6 \]

For 32x32, \sim 310\text{days}
How to measure *all* the correlations?

**Hard way:** Measure the coincidences coming from every pair of pixels in transverse position and momentum planes.

**Medium way:** Measure coincidences coming from pairs expected to be correlated, and local neighborhood.

(Nearest K Neighbors)

\[ M \sim k \times n^2 \quad T \sim k \, n^4 \]

For 32x32, \( \sim 14 \text{ - } 15 \text{ days (for } k \sim 49) \)
How to measure all the correlations?

**Hard way:** Measure the coincidences coming from every pair of pixels in transverse position and momentum planes.

**Medium way:** Measure coincidences coming from pairs expected to be correlated, and local neighborhood.

**Easy way:** Measure coincidences coming from random patterns and reconstruct whole thing using Compressive Sensing Tomography.

\[
M \sim n^2 \log(n) \quad T \sim n^4 \log(n)
\]

For 32x32, \(~ 8 \text{ hours}\)
Why measure these correlations?

With strong enough correlations,
In complementary domains,
you can prove there’s entanglement
by way of \textbf{EPR-steering}!

\[
\Delta(x_A) \cdot \Delta(k_A) \geq \frac{1}{2}
\]

But...

\[
\Delta(x_A|x_B) \cdot \Delta(k_A|k_B) \geq 0
\]
EPR-steering: Essential Concepts

- **The situation**: Alice and Bob share a separated pair of particles A and B entangled in (e.g.,) position and momentum.
- **Locality**: The effect of measurement cannot travel faster than light.
- **Completeness**: Quantum Mechanics gives a complete description of reality. The uncertainty principle is an absolute fundamental limit.
- **The steering**: Alice’s choice of measurement controls the ensemble of possible states Bob measures.
From the EPR-paradox to EPR-steering

• **Locality:**
Everything about particle B is in information $\lambda$ in B’s past light cone.
  - Conditioning on Alice’s results cannot reduce the uncertainty more than conditioning on all of $\lambda$
    \[ \Delta(x_B | x_A) \geq \Delta(x_B | \lambda) \]

• **Completeness:**
The uncertainty principle still holds, even when conditioning on all this information $\lambda$.
    \[ \Delta(x_B | \lambda) \cdot \Delta(k_B | \lambda) \geq \frac{1}{2} \]

• Putting Locality and completeness together:
We get the first EPR-steering inequality (Reid, 1989)
    \[ \Delta(x_B | x_A) \cdot \Delta(k_B | k_A) \geq \frac{1}{2} \]

• But in general, we know that:
    \[ \Delta(x_B | x_A) \cdot \Delta(k_B | k_A) \geq 0 \]

**The EPR Paradox:**
Locality and Completeness are mutually exclusive.
The Flavors of EPR-steering

- **Easy**: Show the joint entangled state is EPR-steerable
- **Medium**: Demonstrate EPR-steering correlations (i.e., the EPR paradox)
  - Useful in more robust Quantum Key Distribution
- **Hard**: “EPR-steer” something
  - Actually do the tomography on Bob’s systems to show that Bob’s state conditioned on Alice’s measurement result is under her control
Proving EPR-steering (Theoretically)

- Lots of correlations can be explained classically
  - Particles A and B could be classically correlated to begin with
  - Alice or Bob could have an untrusted measurement device (a “black box”)

- But if Bob was not receiving halves of entangled pairs...
  - Then there’s a limit to how well Alice can predict Bob’s measurement results.

\[
h(x) \equiv -\int dx \rho(x) \log(\rho(x))
\]
\[
h(x_B | x_A) \geq \int d\lambda \rho(\lambda) h(x_B | \lambda)
\]
\[
h(k_B | k_A) \geq \int d\lambda \rho(\lambda) h(k_B | \lambda)
\]
\[
h(x_B | \lambda) + h(k_B | \lambda) \geq \log(\pi e)
\]

(Walborn et al., 2011)
Proving EPR-steering (Experimentally)

\[ h(x_B | x_A) + h(k_B | k_A) \geq \log(\pi e) \]
- This relies on knowing continuous probability densities

But...

- Discrete approximation never decreases entropy!*

\[ H(X_B | X_A) + \log(\Delta x_B) \geq h(x_B | x_A) \]

- So violating the discrete inequality...

\[ H(X_B | X_A) + H(K_B | K_A) \geq \log\left(\frac{\pi e}{\Delta x_B \Delta k_B}\right) \]

- ...witnesses position-momentum EPR-steering!

*See PRL 110, 130407 (2013) for full treatment

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Figure from: Phys. Rev. X, 3, 011013 (2013).
Application: Monogamy of EPR-steering Correlations

• A and B’s (sufficiently) high correlations can guarantee low correlations with any third party
  – (good for security against eavesdroppers)

• Steering deficit: \( \delta_{A \rightarrow B} \)

\[
\delta_{A \rightarrow B} \equiv H(X_B | X_A) + H(K_B | K_A) - \log \left( \frac{\pi e}{\Delta x_B \Delta k_B} \right)
\]

• Monogamy inequality:

\[
\delta_{A \rightarrow B} + \delta_{C \rightarrow B} \geq 0
\]
(made by combining two uncertainty relations)

• Security:

Alice and Bob can use their higher correlations to distill a secret key for private communication.
Thanks for Listening!
Works Cited


Entanglement and the Hierarchy of Locality

• Local Hidden Variables (LHV):
  – Information existing in past light cone
  – LHV Models:
    \[ \rho(x_A, x_B) = \int d\lambda \rho(\lambda)\rho(x_A|\lambda)\rho(x_B|\lambda) \]
    – Ruled out by Violating a Bell Inequality

• Local Hidden States (LHS):
  – States determined by local hidden variables

• LHS model (for Bob):
  \[ \rho(x_A, x_B) = \int d\lambda \rho(\lambda)\rho(x_A|\lambda)\text{Tr}\left[\hat{\Pi}_X^B \hat{\rho}_B^B\right] \]
    – Ruled out by violating an EPR-steering inequality.

• Separable model:
  \[ \rho(x_A, x_B) = \int d\lambda \rho(\lambda)\text{Tr}\left[\hat{\Pi}_X^A \hat{\rho}_A^A\right]\text{Tr}\left[\hat{\Pi}_X^B \hat{\rho}_B^B\right] \]
    – Ruled out by any entanglement witness.