CHAPTER 10 (CONT) MORE ON TORQUE

MORE ON THE VECTOR NATURE OF TORQUE (AND $\vec{\omega}$ AND $\vec{\alpha}$)

The direction of the vector representing $\vec{\omega}$ and $\vec{\alpha}$ is determined by the right hand rule:

Curl the fingers of your right hand around the rotation axis in the direction of rotation. Then, your thumb points in the direction of the vector.

The torque is also a vector.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Recall, since $\vec{\tau}$ is a cross product, it is orthogonal to $\vec{r}$ and $\vec{F}$.

Vector from the rotation axis to the point where the force is applied.

Example: Pulley with tension

To what direction is the torque vector pointing?

Redraw both vectors, pivot as originating from the same point. Apply right hand rule: $\vec{\tau}$ points out of paper.
A static view:

\[
\text{Length of torque vector is determined by:}
\]

\[
|\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta
\]

Since we know that

\[
\mathbf{\omega} \times \mathbf{r} = \mathbf{\omega}
\]

we know that \( \mathbf{\omega} \) points in the same direction as torque.

This means that \( \frac{d\omega}{dt} \) (and \( d\omega \)) also points in the same direction as torque.

**Rotational Energy**

For linear motion we saw that translational kinetic energy was given by

\[
KE_t = \frac{1}{2} m v^2
\]

Similarly, for rotating objects, there is a kinetic energy associated with rotation

\[
KE_R = \frac{1}{2} I \omega^2
\]

Since \( KE_R = \frac{1}{2} m v^2 = \frac{1}{2} m (ru)^2 = \frac{1}{2} (mr^2) \omega^2 = \frac{1}{2} I \omega^2 \) (for a point mass)
**Example** A rod swings from an initial horizontal position. What is its angular velocity as it passes through a vertical orientation?

Treat as a conservation of energy problem.

\[ E_i = mgh \]
\[ E_f = KE_r + U_g \]

But what are \( h\), \( KE_r \), \( U_g \)?

We can choose \( U_g = 0 \), then \( h \) would be the relative height of the CM of the system from zero to the initial state.

Thus \( h = \frac{L}{2} \)

So \( E_i = mg \left( \frac{L}{2} \right) = \frac{1}{2} I \omega^2 = E_f \)

\[ I = \frac{1}{3} ML^2 \]

\[ \frac{\omega^2}{\frac{1}{3} ML^2} = \frac{\frac{3g}{L}}{\frac{1}{2}} = \frac{3g}{2} \]

\[ \omega = \sqrt{\frac{3g}{2}} = \sqrt{12.2} \text{ rad/s} = \omega \]
WHAT THEN IS THE VELOCITY OF THE TIP AT THIS POSITION?

\[ v_{\text{tip}} = 2.4 \text{ m/s} \]

NOTE THAT IF THE ROD WERE DROPPED FROM HEIGHT \( L \), IT WOULD ONLY HAVE VELOCITY \( \frac{1}{2} m v^2 = mg L \Rightarrow v = \sqrt{2gL} \leq \sqrt{3gL} \) SO THE TIP IS MOVING FASTER THAN THE ROD WOULD BE IF IT WERE SIMPLY DROPPED!

THE WORK-ENERGY THEOREM

As before, \( W_{\text{net}} = \Delta K.E. \)

Now, \( \Delta K.E. = \Delta K_{\text{tip}} + \Delta K_{\text{rod}} \)

Work from a torque is found by starting from our old definition

\[ W = \int F \cdot dl \]

= \[ \int F_r \theta d\theta \]

\( F_r \) is the component of force parallel to \( dl \) \( \theta \) is the component perpendicular to \( r \)

\[ F_r = \tau \]

\[ W = \int \tau d\theta \]

or \( W = \tau \Delta \theta \) for constant \( \tau \)
EXAMPLE: TWO CYLINDERS, ONE SOLID, ONE HOLLOW, EACH HAVE R = R₀, MASS M. IF THEY ARE HELD AT THE TOP OF AN INCLINED PLANE (HEIGHT h)

WHAT ARE THEIR TRANSLATIONAL SPEEDS AT THE BOTTOM OF THE RAMP? HOW DOES THIS COMPARE TO A BLOCK OF MASS M? (ASSUME NO SLIPPING FOR THE CYLINDERS AND NO FRICTION FOR THE BLOCK)

HOLLOW (I = MR²)

\[ E_i = Mgh \]
\[ E_f = \frac{1}{2} M v_f^2 + \frac{1}{2} I \omega^2 \]

\[ Mgh = \frac{1}{2} M v_f^2 + \frac{1}{2} MR^2 \omega^2 \]

BUT \( R^2 \omega^2 = v^2 \)

SO \( Mgh = Mv^2 \rightarrow v_f = \sqrt{2gh} \)

SOLID (I = \frac{1}{2} MR²)

\[ E_i = Mgh \]
\[ E_f = \frac{1}{2} M v_f^2 + \frac{1}{2} I \omega^2 \]

\[ Mgh = \frac{1}{2} M v_f^2 + \frac{1}{2} \frac{1}{2} MR^2 \omega^2 \]

\[ Mgh = \frac{1}{2} M v_f^2 + \frac{1}{4} MR^2 \omega^2 \]

\[ Mgh = \frac{1}{2} M v_f^2 + \frac{1}{4} M v^2 \rightarrow v_f = \sqrt{3gh} \]
So, $v_S > v_H$

For a block (No Rotation)

$$Mgh = \frac{1}{2} MV^2 \quad V = \sqrt{2gh}$$

So, $v_B > v_S > v_H$

The larger $I$ is, the larger the ratio $\frac{KE}{KE_T}$ is, so the smaller $V$ is.