FINAL REVIEW

ROTATIONAL KINEMATICS

For a circle of radius R

\[ S = \Theta R \quad \text{is a length of arc on the circumference subtended by an angle } \Theta \ (in \ radians) \]

\[ \Theta \] is a useful coordinate. It measures the azimuthal position relative to some axis.

Angular Vel. \[ \vec{\omega} = \frac{\Delta \Theta}{\Delta t} \]

Angular Acc. \[ \vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t} = \frac{\partial \vec{\omega}}{\partial t} \]

If \( \alpha \) is constant our old kinematic equations are valid:

\[ \Theta_f = \Theta_i + \omega_i t + \frac{1}{2} \alpha t^2 \]

\[ \omega_f = \omega_i + \alpha t \]

\[ \omega_f^2 - \omega_i^2 = 2 \alpha (\Theta_f - \Theta_i) \]

And since \( \Theta = \frac{S}{R} \) we know that at the edge (R)

\[ V = \omega R \quad \text{and} \quad a = \alpha R \quad \text{are the linear (tangential) velocity and acceleration.} \]

Finally, \( S = \frac{\omega}{2\pi} \), \( T = \frac{1}{2} S = \frac{\pi \omega}{2} \)
Torsion

A force applied to an object at a position \( r \) relative to an axis the object can pivot around creates a torque according to

\[ \tau = r \times F = |r| |F| \sin \theta \hat{\tau} \quad \text{[N m]} \]

\( \hat{\tau} \) given by right hand rule.

Just as forces cause mass to accelerate, torques cause them to rotationally accelerate according to

\[ \Sigma \tau = I \ddot{\theta} \quad I \text{ is moment of inertia's second law for torques} \]

\[ I = \int r^2 \, dm \quad [I] = \text{kg m}^2 \]

or \[ I = \sum_i m_i \, r_i^2 \]

Torques "act" like forces in rotational systems. Moments of inertia "act" like mass.

For moments of inertia about axes not going through an object's center of mass, you can use the parallel axis theorem.

\[ I = I_{cm} + m \, h^2 \]

\( h \) is distance between axes.

Moment of inertia around parallel axes that passes through CM.
GOOD TO KNOW

SPHERE: \( I = \frac{2}{5} \pi m r^2 \)

DISC/CYLINDER: \( I = \frac{1}{2} \pi m r^2 \)

BAR: \( I_{cm} = \frac{1}{12} m L^2 \) \( I_{total} = \frac{1}{3} m L^2 \)

MASS DENSITY

\( \lambda = \frac{M}{L} \) \( \sigma = \frac{M}{A} \) \( \rho = \frac{M}{V} \)

SOLVING TORQUE PROBLEMS

- DRAW PICTURES
- DRAW FORCE DIAGRAM (\( \vec{F} \))
- DRAW FORCES WHERE they ACT
- CHOOSE COORDINATES
- SUM FORCES AND TORQUES

\( \sum F = ma \) \( \sum \vec{r} \times \vec{F} = \tau \)

- IDENTIFY UNKNOWNS AND ELEMENTARY TURNS

ROTATIONAL ENERGY / WORK

\( KE = \frac{1}{2} I \omega^2 \)

\( \omega = \int \tau \, d\theta = \tau \Delta \theta \)

\( \tau \) FOR CONSTANT TORQUE

AS \( \tau \) \( \omega \)

\( E = KE + U + \text{[other forms]} \) \( = \text{constant} \).
Angular Momentum

\[ \mathbf{L} = I \mathbf{\omega} = \mathbf{r} \times \mathbf{p} \]

Like force, torque was analogous to force, angular momentum is analogous to linear momentum.

Since \( \mathbf{L} = \mathbf{I} \mathbf{\omega} \) and \( \mathbf{\alpha} = \frac{d \mathbf{\omega}}{dt} \)

Then \( \mathbf{L} \cdot \mathbf{\alpha} = \frac{d \mathbf{L}}{dt} \)

For systems on which no external torques are acting, angular momentum is constant.

For objects that have angular momentum and are subject to torques that are perpendicular to the existing \( \mathbf{L} \), precession will occur.

Statics

Sum torques and forces, net of each must be zero.

\[ \sum \mathbf{\tau} = 0 \quad \text{for all axes} \]

\[ \sum \mathbf{F} = 0 \quad \text{for all directions} \]
FLUIDS

DENSITY \[ \rho = \frac{M}{V} \quad (M = \rho V) \]

WEIGHT OF FLUID \[ mg = \rho V g \]

PRESSURE \[ P = \frac{F}{A} \quad [P] = \frac{N}{m^2} = Pa \]

- Pressure is a scalar and extended in all directions.
- In a static fluid, pressure is constant at a constant depth.

\[ P = P_0 + \rho gh \quad \text{where} \quad h = \text{depth} \]

AND \[ P_0 = 101,300 \quad \text{N/m}^2 \]

For most pressure gauges measure relative pressure (versus atmospheric).

So \[ P = P_{\text{gauge}} + P_0 \]

Pascal's Principle

In a closed system an applied pressure is added to all points in the fluid so,

\[ \frac{F_{\text{in}}}{A_{\text{in}}} = \frac{F_{\text{out}}}{A_{\text{out}}} \quad \Rightarrow \quad F_{\text{out}} = F_{\text{in}} \left( \frac{A_{\text{out}}}{A_{\text{in}}} \right) \]
Bouyancy (Archimedes) Principle

For an object which displaces some volume of fluid, feels an upward force equal to the weight of the fluid displaced

\[ F_B = \rho g V \]

Conservation of Mass (Continuity Eqn)

\[ \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \]

For most liquids (water included), density is constant (i.e., incompressible).

\[ \rho_1 V_1 A_1 = \rho_2 V_2 A_2 \] (when density is constant)

Bernoulli's Equation

\[ P_1 + \frac{1}{2} \rho_1 V_1^2 + \rho_1 g y_1 = P_2 + \frac{1}{2} \rho_2 V_2^2 + \rho_2 g y_2 \]

\[ \rightarrow \text{For constant height, } \rho_2 \text{ constant, } P \text{ increases as } V \text{ decreases.} \]

\[ \rightarrow \text{For constant speed, } P \text{ increases as } h \text{ decreases.} \]
OSCILLATION

Since \( F = ma = m \frac{d^2x}{dt^2} \)

Any system for which this position satisfies

\[ \frac{d^2x}{dt^2} + \omega_0^2 x = 0 \]

is said to oscillate with "simple harmonic motion."

A solution to this equation is

\[ x(t) = A \cos(\omega_0 t + \phi) \quad A = \text{amplitude} \]

\[ \phi = \text{phase} \]

and \( A, \phi \) are free parameters which will change depending on initial conditions.

For springs:

\[ F = -kx = m \frac{d^2x}{dt^2} \]

\[ \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \]

So \( \omega_0 = \sqrt{\frac{k}{m}} \) for a spring.

For a simple pendulum:

\[ F = -mg \sin \theta \quad (\theta \text{ is angle from vertical}) \]

So

\[ -mg \sin \theta = m \frac{d^2\theta}{dt^2} = m l \frac{d^2\theta}{dt^2} \]

\[ \frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0 \quad \text{if } \theta \text{ is small, } \sin \theta \approx \theta \]

So \( \frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0 \)

\[ \Rightarrow \omega_0 = \sqrt{\frac{g}{l}} \]
Waves can be transverse or longitudinal.

Characterized by amplitude, wavelength ($\lambda$) and frequency ($f$) (or $\omega$).

Wave velocity $v = \frac{\lambda}{f}$

Where $k = \frac{2\pi}{\lambda}$ is the wave number.

For transverse waves, particles oscillate in a perpendicular direction to propagation.

For longitudinal waves, particles oscillate parallel to propagation.

Wave equations are of the form:

$$y(x,t) = A \sin(kx - \omega t + \phi)$$

Waves add linearly, which allows for constructive and destructive interference.

"Cavities" with fixed length support standing waves of wavelengths $\lambda$:

$$L = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$$

$L$ is cavity length.

Energy in waves/oscillations:

$$E \propto A^2 \cdot \omega^2 \left( \frac{\lambda^2}{2} \right)$$

Energy is proportional to the square of amplitude and square of frequency.
**Sound**

Sound is a compression (longitudinal) wave moving through air (or something else).

\[ V_{\text{sound}} = 343 \text{ m/s} \]

Sound coming from or detected by objects moving through the supporting fluid are shifted in frequency.

If observer/source are moving toward each other

\[ f' = \left( \frac{V_{\text{sound}} + V_{\text{obs.}}}{V_{\text{sound}} - V_{\text{obs.}}} \right) f \quad \text{if moving away from each other} \]

\[ f' = \left( \frac{V_{\text{sound}} - V_{\text{obs.}}}{V_{\text{sound}} + V_{\text{obs.}}} \right) f \]

Sound waves interfere as other waves do.

Waves of different frequency interfere in time and produce "beats" with a frequency

\[ f_{\text{beat}} = |f_1 - f_2| \]

Identical waves originating from different sources interfere in space, due to the different path lengths \( \lambda, \lambda \pm \lambda \),

\[ \frac{|d_1 - d_2|}{\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \] waves will cancel (destructive)

\[ \frac{|d_1 - d_2|}{\lambda} = 0, 1, 2, 3, \ldots \] waves will interfere.