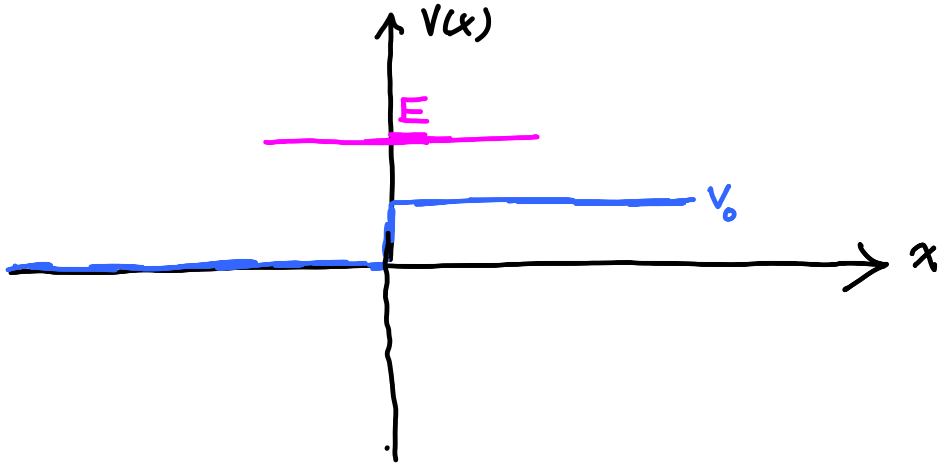
Physics 123 - April 15, 2013

Last Time

Step potential problem



Barrier and step potentials

particle w/ energy E incident on step potential from "left"

$V(x) = 0$ for $x<0$

$V(x) = V_0$ for $x>0$

$E > V_0$

general solns of Schr. Time-independent Eqn

$$\psi(x)\Big|_{x<0} = Ae^{ik_1 x} + Be^{-ik_1 x} \quad \text{where} \quad k_1^2 = \frac{2mE}{\hbar^2}$$

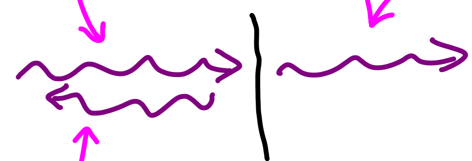
$$\psi(x)\Big|_{x>0} = Ce^{ik_2 x} + De^{-ik_2 x} \quad \text{where} \quad k_2^2 = \frac{2m(E-V_0)}{\hbar^2}$$

Put in Boundary conditions to constrain solution further

I: particle incident on potential from $-\infty$ in this problem.

e^{ik_1x} — incoming particle

e^{ik_2x} — transmitted particle



$x=0$

e^{-ik_1x} — Reflected particle

Have reflected + transmitted waves, but no wave moving right to left from $+\infty$ $\rightarrow$ $D=0$

So

$$\psi(x)\Big|_{x<0} = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\psi(x)\Big|_{x>0} = Ce^{ik_2x}$$

II: $$\psi(0)\Big|_{x<0} = \psi(0)\Big|_{x>0} \quad \rightarrow \quad \boxed{A+B=C}$$

demand wavefunction be continuous at $\underline{\underline{x=0}}$

III: $\left.\dfrac{d\psi(0)}{dx}\right|_{x<0} = \left.\dfrac{d\psi(0)}{dx}\right|_{x>0}$ demand $\frac{d\psi}{dx}$ be continuous at $x=0$

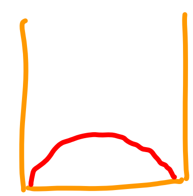
Use this Boundary condition for finite V.
If $V \rightarrow \infty$ $d\psi/dx$ NOT continuous
As we saw for ∞ square well

$$iAk_1 - iBk_1 = ick_2$$

$$(A-B)k_1 = ck_2$$

Left w/ 2 equations and 3 unknowns
Solve for B and C in terms of A, then use Normalization $\int \psi^* \psi = 1$ to get A

$$B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right) A \qquad C = \left(\frac{2k_1}{k_1 + k_2}\right) A$$



for $x \leq 0$ $\quad \psi(x) = A e^{ik_1 x} + A\left(\frac{k_1 - k_2}{k_1 + k_2}\right) e^{-ik_1 x}$

$x \geq 0$ $\quad \psi(x) = A\left(\frac{2k_1}{k_1 + k_2}\right) e^{ik_2 x}$

use $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$ to solve for A . . . Won't do that here . . .

for $x \leq 0$, incoming wave

$x \leq 0$ $\quad \psi(x) = A e^{ik_1 x} + A\left(\frac{k_1 - k_2}{k_1 + k_2}\right) e^{-ik_1 x}$

for $x \leq 0$
Reflected wave

$x \geq 0$ $\quad \psi(x) = A\left(\frac{2k_1}{k_1 + k_2}\right) e^{ik_2 x}$

for $x \geq 0$
Transmitted wave

Probability that particle is in "one of the waves" goes as $\psi^* \psi$

incoming wave $\quad \psi^* \psi \sim A^* e^{-ik_1 x} A e^{ik_1 x} \sim A^2$ for real A

Intensity is velocity weighted probability density

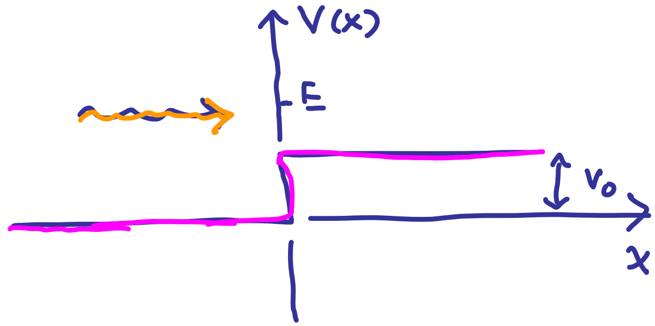
$$I \propto v \psi^* \psi$$

$$v = \frac{p}{m} = \frac{\hbar k}{m}$$

$R \equiv$ reflection Coefficient $\equiv \dfrac{\text{Intensity Reflected beam}}{\text{Intensity incoming beam}} = \dfrac{k_1 B^* B}{k_1 A^* A} = \left(\dfrac{k_1 - k_2}{k_1 + k_2} \right)^2$

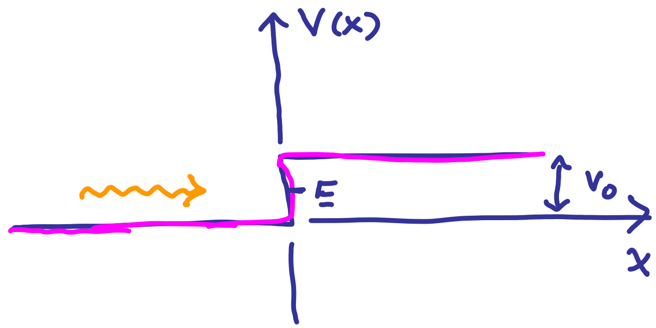
$T \equiv$ Transmission Coefficient $\equiv \dfrac{\text{Intensity Transmitted beam}}{\text{Intensity incoming beam}} = \dfrac{k_2 C^* C}{k_1 A^* A} = \dfrac{4 k_1 k_2}{(k_1 + k_2)^2}$

We just examined



$E > V_0$

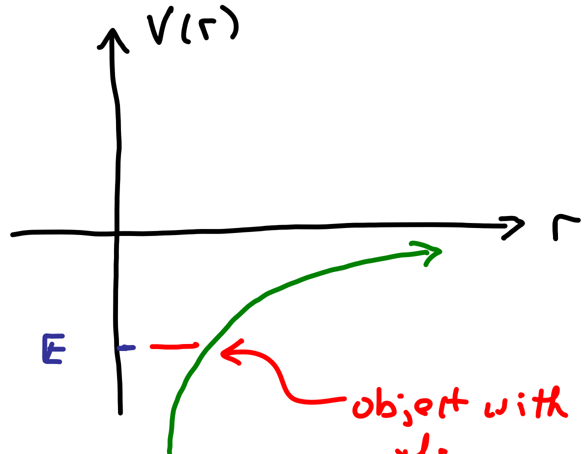
Consider



$E < V_0$

What happens in this situation classically ??

Can you think of a classical example ?

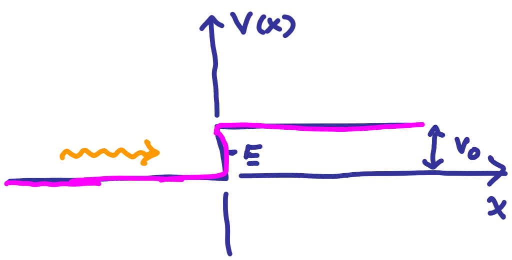


Grav. Potential $\sim -\dfrac{GMm}{r}$

Rocket adds at least Energy $= -\dfrac{GMm}{r}$ $\rightsquigarrow$ Escape velocity

object with this energy is bound

What happens In quantum mechanics



$E < V_0$

To left of step $(x \le 0)$

$$\psi(x) = A e^{ik_1 x} + B e^{-ik_1 x}$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$ Same as before

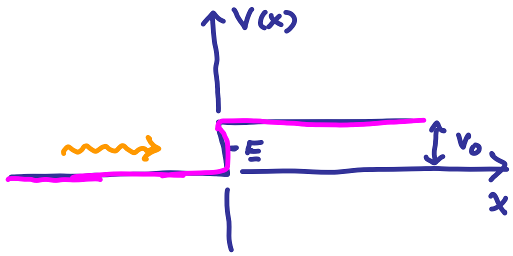
For $x \geq 0$ (classically forbidden region)

$\psi \sim e^{ikx}$ w/ $k = \frac{\sqrt{2M(E-V_0)}}{\hbar}$ does not work since $E - V_0 < 0$

Try $\psi \sim e^{ik'x}$ where $k' = i\frac{\sqrt{2M(V_0-E)}}{\hbar} = ik$

$+\cdots$ so well-defined

So $\psi(x) = C e^{-kx}$ where $k = \frac{\sqrt{2M(V_0-E)}}{\hbar}$ works as a soln (try it)



$E < V_0$

$x \leq 0 \qquad \psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$

$$k_1 = \frac{\sqrt{2ME}}{\hbar}$$

$x \geq 0 \qquad \psi(x) = ce^{k_2x} + De^{-k_2x}$

$$k_2 = \frac{\sqrt{2M(V_0-E)}}{\hbar}$$

B.C. I

$x \to \infty, \; e^{k_2x} \to \infty \quad \rightarrow \quad c = 0$

B.C. II

$\psi(0)\big|_{x\leq 0} = \psi(0)\big|_{x\geq 0} \quad \rightarrow \quad A + B = D$

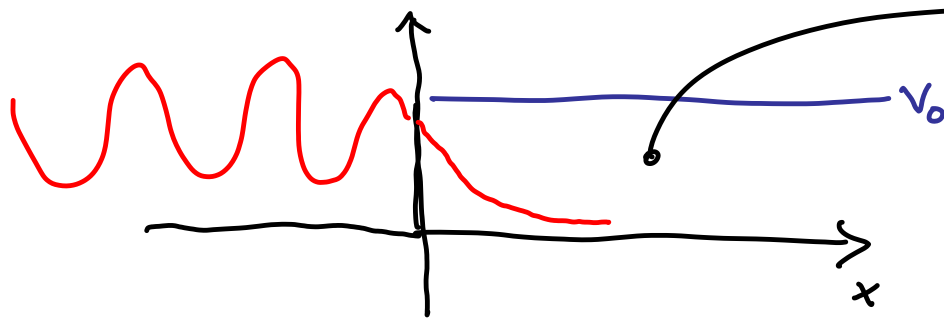
B.C. III $\left.\frac{d\psi(0)}{dx}\right|_{x\le 0} = \left.\frac{d\psi(0)}{dx}\right|_{x\ge 0}$ $ik_1A - ik_1B = -k_2D$

Solve for A, B in terms of D

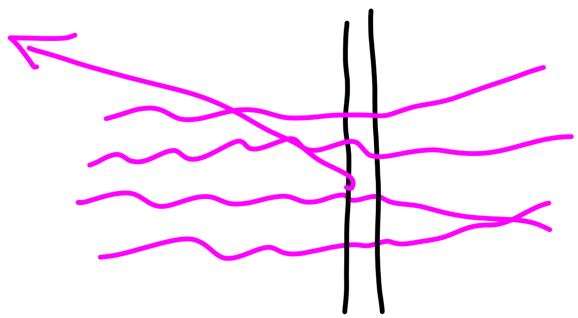
$$A = \frac{D}{2}\left(1 + \frac{ik_2}{k_1}\right)$$

$$B = \frac{D}{2}\left(1 - \frac{ik_2}{k_1}\right)$$

$$\psi(x) = \begin{cases} \frac{D}{2}\left(1 + \frac{ik_2}{k_1}\right)e^{ik_1 x} + \frac{D}{2}\left(1 - \frac{ik_2}{k_1}\right)e^{-ik_1 x} & x \le 0 \\ D e^{-k_2 x} & x \ge 0 \end{cases}$$

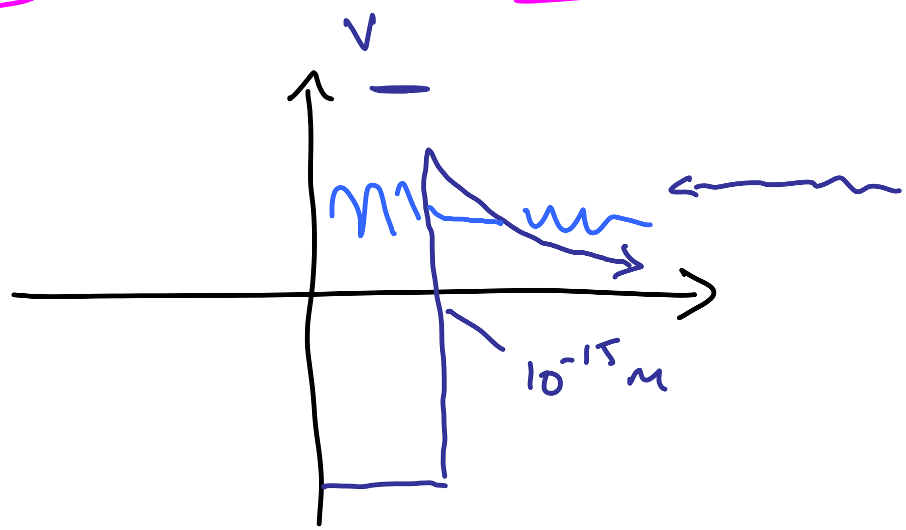


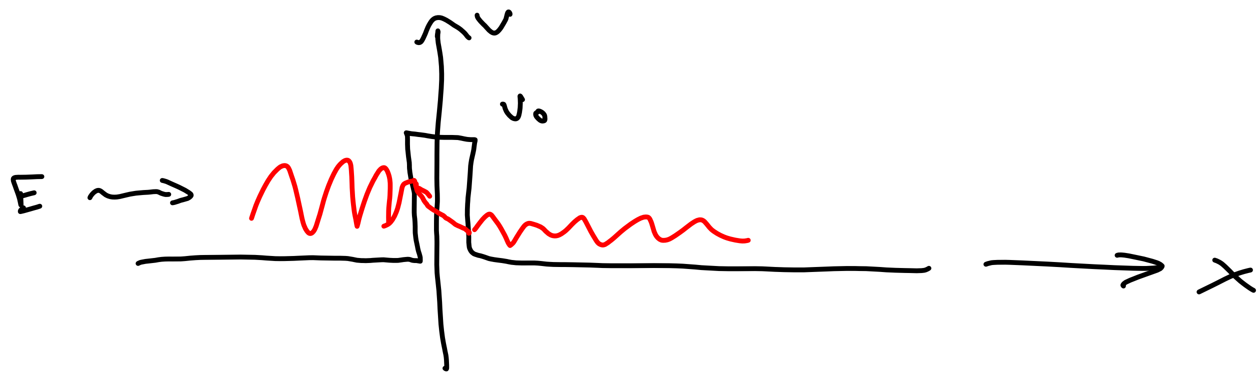
$\psi^* \psi \sim e^{-2k_2 x}$



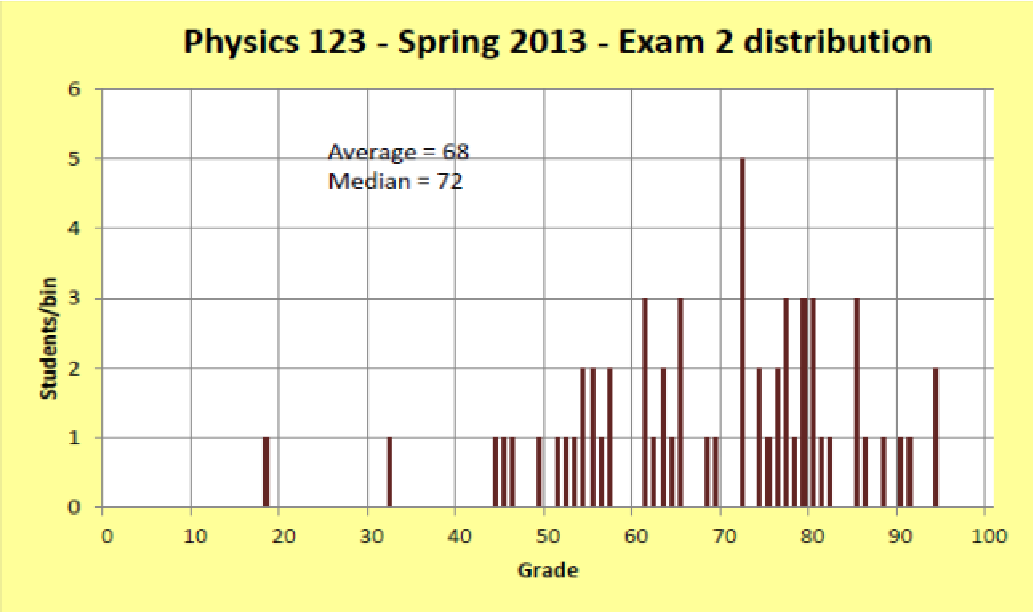
alpha decay

Quantum tunneling!





There is a non-vanishing probability (that depends on the width and height of the potential barrier) that the particle can be found on the other side of the barrier ... in a region that is "classically forbidden". This phenomenon is known as quantum tunneling.


Physics 123 - Spring 2013 - Exam 2 distribution
Average = 68
Median = 72
Students/bin
Grade
0
1
2
3
4
5
6
0
10
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30
40
50
60
70
80
90
100