REFERENCE FRAMES AND THE OBSERVABLE PHASE SHIFT OF THE CARRIER IN G.W. INTERFEROMETRIC DETECTORS

A.C. Melissinos

April 28, 2003

1. Introduction

In general interferometric detectors of g.w. operate with so called "mirrors in free fall"; in practice the mirror is free to move along the relevant axis in the x-y plane. These mirrors define the optical cavity. However one can consider other configurations where the mirrors are fixed or subject to strong restoring forces in the laboratory frame. Bar detectors closely approximate this latter configuration.

One can calculate the optical shift imposed by the g.w. on the carrier in any convenient frame of reference. Of course the observable (the phase shift) must be independent of the frame used in the calculation. Two frames in particular are highly relevant. The first is the frame in which the g.w. is expressed in the TT (transverse traceless) gauge. This is the "free-fall" frame where the distance between geodesics remains fixed. The other frame is that of the "local observer", namely the laboratory frame.

The transfer functions between the g.w. amplitude $h(t)$ and the observable phase shift are often derived by using Laplace transforms. We will work in both the time domain and Laplace domain and, of course, obtain the same results in both cases.

2. Free mirrors

Throughout we will assume that the g.w. is incident along the negative z-axis and that the arms of the interferometer are along the directions of polarization of the wave. We will calculate only for one arm. The g.w. amplitude is written in the form

$$h(t,z) = h_0 \, \Re e\{e^{i(\Omega t + z/c)}\} \tag{1}$$

In general we will also set $z = 0$, and write $h(t)$. The length of the arm is L and we introduce

$$T = L/c \tag{2}$$

In the TT gauge, the metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} \qquad h_{\mu\nu}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t) & h_\times(t) & 0 \\ 0 & h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

with $\eta_{\mu\nu} = (1, -1, -1, -1)$. Recall that free objects follow geodesics, thus in the TT frame the coordinates of free mirrors do not change.

Usually the time for the round trip of the carrier is calculated by appealing to the fact that light propagates on a null geodesic

$$ds^2 = 0 = g_{\mu\nu}\, dx^\mu dx^\nu = c^2 dt^2 - [1 + h(t)]dx^2 \tag{4}$$

In writing Eq.(4) we assumed that $dy = dz = 0, h_x(t) = 0$ and set $h(t) = h_+(t)$. By calculating to first order in h we obtain

$$dx = \left[1 - \frac{1}{2}h(t)\right] cdt \tag{5}$$

To obtain the total distance traversed by the light we integrate dx from 0 to L and $-dx$ from L back to 0. For $h(t)$ we use Eq.(1) with $z = 0$ so that

$$\Delta X\left(t = \frac{2L}{c}\right) = c\int_0^{L/c}\left(1 - \frac{h_0}{2}e^{i\Omega t}\right)dt + c\int_{L/c}^{2L/c}\left(1 - \frac{h_0}{2}e^{i\Omega t}\right)dt \tag{6}$$

In our approximation using L/c and $2L/c$ for the limits in the integration is perfectly consistent. Thus

$$\Delta X(t) = 2L - h(t)L\frac{\sin\Omega T}{\Omega T}e^{-i\Omega T} \tag{7}$$

Here we shifted back in time from $\Delta X(2L/c)$ to $\Delta x(t)$ by multiplying by $\exp[i(\Omega t - 2\Omega T)]$.

We are interested only in the time dependent part of the phase shift, $\Delta\phi(t) = k\Delta x(t)$ with k the wavevector of the carrier, $k = \omega_c/c$. Thus

$$\Delta\phi(t) = \omega_c T\ h(t)\frac{\sin\Omega T}{\Omega T}e^{-i\Omega T} \tag{8}$$

This is the desired result. We will rederive it, in the local observer's frame in the following section.

3. The local observer's frame

The results that we give follow the derivation of Pegoraro et al [1] even though similar results have been reported by Rakhmanov [2] who cites Grishchuk [3,4].

We introduce a general coordinate transformation that depends on a parameter λ such that

$$\begin{aligned} x^0(\lambda_2) &= x^0(\lambda_1) + \frac{1}{2c}(\lambda_2 - \lambda_1)x^i \ \dot{h}_{ij} \ x^j \\ x^i(\lambda_2) &= x^i(\lambda_1) - (\lambda_2 - \lambda_1)h^i_j x^i \\ x^3(\lambda_2) &= x^3(\lambda_1) + \frac{1}{2c}(\lambda_2 - \lambda_1)x^i \ \dot{h}_{ij} \ x^j \end{aligned} \tag{9}$$

with $i, j = 1, 2.$ and $\dot{h} = \partial h/\partial t$. Under this transformation the metric tensor takes the form

$$h_{\mu\nu}(\lambda) = (2\lambda - 1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{21} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{\lambda}{c^2}\left[x^i(\lambda) \ \ddot{h}_{ij} \ x^j(\lambda)\right]\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \tag{10}$$

The parameter λ is related to the mechanical properties of the detector through

$$\lambda = \frac{1}{2}\frac{\omega_m^2}{(\omega_m^2 - \Omega^2)} \tag{11}$$

Here we have modelled the detector end walls (i.e. the mirrors) as a simple harmonic oscillator of resonant frequency ω_m and damping $\gamma_m(Q_m = \omega_m/\gamma_m)$. As before Ω is the g.w. frequency. From Eq.(10) we see that $\lambda = 0$ corresponds to the TT frame and is obtained when $\Omega \gg \omega_m$. The case $\lambda = 1/2$ corresponds to the local frame and is obtained when $\Omega \ll \omega_m$. In the special case that $\Omega = \omega_m$, namely when the mechanical properties of the detector are resonant with the g.w. we must account for the damping and

$$\lambda = Q_m = \omega_m/\gamma_m \tag{12}$$

In the local frame <u>free</u> mirrors <u>move</u> under the influence of the gravitational wave as can be seen from Eq.(9). Set $\lambda_1 = 0, \lambda_2 = 1/2$, to find

$$x'^i = x^i - \frac{1}{2} h^i_j x^j$$

where the prime refers to the coordinates in the local frame. The unprimed coordinates are in the TT frame and remain constant for free mirrors. This is the usual explanation for the observable phase shift, but it is strictly true only in the limit $\Omega T \to 0$. This is because there is an additional contribution to the phase shift. In the local frame, as can be seen from Eq.(10), h_{00} is space and time dependent and this is equivalent to a gravitational potential. As a result the frequency of the carrier is shifted and this contributes to the phase shift as well [2].

The equivalent potential for $\lambda = 1/2, h_+ \neq 0, h_\times = 0$ is

$$\Phi = \frac{1}{2} h_{00} c^2 = \frac{1}{4}(x^2 - y^2)\ \ddot{h}\ (t, z) \tag{13}$$

and therefore the frequency shift [5]

$$\frac{dk}{k} = -\frac{\Phi}{c^2} \tag{14}$$

Integration over the round trip in a single arm along the x-direction gives a phase shift

$$\eta(t) = \frac{k}{4c^2} \int_0^L \left[\ \ddot{h}\left(t - 2T + \frac{x}{c}\right) +\ \ddot{h}\left(t - \frac{x}{c}\right)\right] x^2 dx \tag{15}$$

Using $h(t)$ as in Eq.(1) the integral is

$$\eta(t) = h_0\ kL\ e^{i\Omega(t-T)} \left[\frac{\sin(\Omega T)}{\Omega T} - 1\right] \tag{16}$$

We must also calculate the phase shift arising from the motion of the free mirrors in the local frame. This is simply

$$\Delta x(t) = 2L\frac{1}{2}h(t - T)$$

or a phase shift

$$\xi(t) = h_0 \; kL \; e^{i\Omega(t-T)} \tag{17}$$

Adding Eqs.(17 and 16) we regain Eq.(8) as we must. Note that in the local frame the i, j elements of the metric tensor are exactly $g_{ii} = g_{jj} = -1$ and $g_{ij} = 0$ for $i \neq j$.

4. Fixed mirrors

This case is most easily analyzed in the local frame where if the mirrors are fixed, their coordinates do not change. The distance between two particles with coordinates x^i and $x^i + \Delta x^i$, is

$$\Delta\ell^2 = \Delta x^i[\eta_{ij} + (2\lambda - 1)h_{ij}]\Delta x^j \tag{18}$$

where we used the metric of Eq.(10) and of course $\Delta x^3 = 0$. Thus, when $\lambda = 1/2$ (local frame) and Δx^i remains fixed, so does $\Delta\ell^2$. The mirrors do not move.

Therefore the only contribution to the phase shift is from the potential induced by the g.w. as given by Eq.(16). In the limit $\Omega t \ll 1$ we obtain

$$\Delta\phi(t) = -h(t)kL\frac{(\Omega T)^2}{6}e^{-i\Omega T} \qquad \begin{array}{c}\text{fixed mirrors}\\ \Omega T \ll 1\end{array} \tag{19}$$

This result should be compared with the shift of free mirrors, Eq.(8), in the same limit

$$\Delta\phi(t) = h(t)kLe^{-i\Omega T} \qquad \begin{array}{c}\text{free mirrors}\\ \Omega T \ll 1\end{array} \tag{20}$$

We see that in this limit the phase shift induced on fixed mirrors is of order $(\Omega T)^2$ (i.e. much smaller) than for free mirrors.

In contrast for $\Omega T \gtrsim \pi$ the response of fixed mirrors tends to

$$\Delta\phi(t) \sim -h(t)kLe^{-i\Omega T} \qquad \begin{array}{c}\text{fixed mirrors}\\ \Omega T \gtrsim \pi\end{array} \tag{21}$$

This dominates over the response of free mirrors which tends to zero in the limit $\Omega T \gg 1$.

5. Laplace domain

The same results can be obtained in the Laplace domain as shown in [2,5]. We will designate the Laplace transform of a function by an overbar

$$\mathcal{L}[f(t)] = \overline{f}(s) = \int_0^\infty e^{-st} f(t) dt \tag{22}$$

Later we will identify $s = i\Omega$ with Ω the g.w. frequency. With this identification, Eq.(22) is a Fourier transform over positive frequencies and the Laplace transform represents the frequency response of the detector.

We start in the TT frame where a round trip in an arm of length L yields a phase shift

$$\phi(t) = \frac{\omega_c 2L}{c} - \frac{\omega_c}{2} \int_{t-2L/c}^{t} h(t')dt' \tag{23}$$

Taking Laplace transforms of both sides we find the time dependent part

$$\begin{aligned} \overline{\Delta\phi}(s) &= \frac{\omega_c}{2} \left\{ \mathcal{L}\left[\int_0^t h(t')dt' \right] - \mathcal{L}\left[\int_0^{t-2T} h(t')dt' \right] \right\} = \\ &= \frac{\omega_c}{2} \left[\frac{\overline{h}(s)}{s} - \frac{e^{-2sT}\overline{h}(s)}{s} \right] = \frac{\omega_c}{2} \frac{1 - e^{-2sT}}{s} \overline{h}(s) \end{aligned} \tag{24}$$

$$\tag{25}$$

By setting $s = i\Omega$ we recover Eq.(8).

We can obtain the same result in the local frame. Since the mirrors are free we must calculate both the effect of mirror motion as well as the frequency shift. Let mirror $\underline{a}$ be at $x_a = 0$ and mirror $\underline{b}$ at $x_b = L$; light leaves mirror $\underline{a}$ at $t = 0$ and reaches $\underline{b}$ at $t = T$, returning to $\underline{a}$ at $t = 2T$. Thus the distance travelled by the light is

$$X(t) = 2L + 2L\frac{1}{2}h(t - T) \tag{26}$$

as also derived before. Taking Laplace transforms of the time dependent part leads to

$$\xi(s) = kLe^{-sT}\overline{h}(s) \tag{27}$$

The Laplace transform for the effect of the gravitational potential can be found by similar methods [2] and is

$$\overline{\eta}(s) = kL \left[\frac{1 - e^{-sT}}{2sT} - e^{-sT} \right] \overline{h}(s) \tag{28}$$

Adding the two contributions from Eqs.(26,27) we regain the transfer function of Eq.(24), which was derived in the TT frame. Setting $s = i\Omega$ in Eqs.(26,27) reproduces Eqs.(17,16).

6. Change in the dielectric tensor

It is well known that the presence of a gravitational field affects the electric and magnetic permeability of the vacuum and thus the velocity of propagation of electromagnetic waves (in vacuum). As a result for fixed frequency the wave vector of the carrier is modified and this leads to an observable phase shift.

As shown in [1] the dielectric tensor is given by

$$\begin{aligned} \epsilon^{ij} &= -[\eta^{ij} - (2\lambda - 1)h^{ij}] \\ \mu_{ij} &= -[\eta_{ij} - (2\lambda - 1)h_{ij}] \end{aligned} \tag{29}$$

In the TT frame ($\lambda = 0$) we also have $\epsilon^{33} = \mu_{33} = -1$. Thus the product of the electric and magnetic permeabilities for propagation in the x-y plane [where one field is polarized in the plane and the other field along the 3-axis] is

$$\epsilon\mu = 1 + h(t)$$

Thus

$$c' = \frac{c}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{1 + h(t)}} \simeq c[1 - \frac{1}{2}h(t)] \tag{30}$$

This is the same result as we obtained in Eq.(5) where $c' = dx/dt$, by directly using the metric tensor. Thus Eqs.(28) represent a completely equivalent description of the effect of the g.w. on the stored carrier light.

In the local frame $(\lambda = 1/2)\epsilon^{ij} = \delta^{ij}(i, j = 1, 2)$ and similarly for μ_{ij}. However ϵ^{33} and μ_{33} are different from unity as can be inferred from Eq.(10). We can write

$$\epsilon^{33} = \mu_{33} = 1 \left[\eta_{33} - \frac{1}{2c^2} x^i \ \ddot{h}_{ij} \ x^j \right] \tag{31}$$

For fixed mirrors this is the only contribution in the local frame and leads to the phase shift given by Eq.(16).

7. Sidebands vs. phase shift

It is well known that a time-dependent phase shift of the carrier is mathematically equivalent (in the frequency domain) to the appearance of sidebands. For instance using the result of Eq.(8) we see that the carrier amplitude $\exp(i\omega t)$ is multiplied by a term of the form

$$e^{i\Delta\phi(t)} = e^{ix\cos\Omega t} \ ; \ x = \omega_c T h_0 \frac{\sin\Omega T}{\Omega T} \tag{32}$$

The exponential can be expanded

$$e^{ix\cos\Omega t} = J_0(x) + iJ_1(x)e^{i\Omega t} + iJ_1(x)e^{-i\Omega t} + ... \tag{33}$$

where we retained only the lowest term in x. If we retard the phase of Ω by $\pi/2$ and expand $J_1(x)$, Eq.(31) can be written in the simpler form as

$$e^{ix\sin\Omega t} = J_0(x) + xe^{i\Omega t} - xe^{-i\Omega t} \tag{34}$$

Thus from an experimental point of view one can either look for a change in the interference pattern as a function of time or for the presence of sidebands in the frequency spectrum of the returned carrier. Which method is used depends on the specific source of g.w. that is sought, and on signal to noise considerations.

8. The case of a long fiber

We can treat the fiber as rigid, i.e. use the expressions for the fixed mirror case. The fiber could be folded so that input and output are at the same spatial coordinate making possible phase shift measurements by interfering part of the input beam with the output. Conversely a long one-way fiber can be used if one is only interested in searching for sidebands.

We first consider a fiber that is coiled with radius R and of total length $L = 2n\ell$ where $\ell = \pi R$ and n is the number of turns in the coil. To directly use Eq.(16) we treat the coil as flattened so that propagation is only along the x-direction with an arm of length ℓ; then n is the number of round trips in the arm. The total phase shift is obtained by the sum of terms given by Eq.(16) with the appropriate delay taken into consideration,

$$\begin{aligned} \eta(t) &= h_0 k\ell \left[\frac{\sin(\Omega\tau)}{\Omega\tau} - 1\right] \sum_{m=0}^{n-1} e^{-2i\Omega\tau m}\, e^{i\Omega(t-\tau)} \\ &= h(t) k \frac{L}{2}\, e^{-i\Omega\tau} \left[\frac{\sin(\Omega\tau)}{\Omega\tau} - 1\right] \frac{\sin(\Omega\tau n)}{n\sin(\Omega\tau)} \end{aligned} \tag{35}$$

where $\tau = \ell/c$.

From physical arguments we expect $\Omega\tau \ll 1$, and if $\Omega\tau n \lesssim \pi/2$ expanding Eq.(35) we find

$$\eta(t) = h(t) k L\, e^{-i\Omega\tau} \frac{(\Omega\tau)^2}{12} \tag{36}$$

which is a "gain" in phase shift over that for a single arm of length ℓ by a factor of $L/2\ell$.

Of more interest is a long fiber of length L in the x-direction. In this case we can take over Eq.(16) directly but we can now assume that $\Omega\tau > \pi$ leading to the result of Eq.(21). As an example if we consider a communications fiber laid out over 1,500 km then the above condition is satisfied for $\nu_g \gtrsim$ 100 Hz, which is in the range of frequencies of interest. One could then consider looking for sidebands on the transmitted light. For $h_0 = 10^{-21}$ the modulation index $x = 10^{-8}$. Thus the sideband is at -160 db with respect to the carrier and at a 100 Hz offset. Whether that is detectable is an issue of noise calculation and signal duration.

References and notes

1. F. Pegoraro, E. Picasso and L.A. Radicati, J. Phys. A 11, 1949 (1978).

2. M. Rakhmanov "Dynamics of Laser Interferometric Gravitational Detectors", California Institute of Technology Thesis, 2000.

3. L. Grishchuk, "Gravitational waves in the cosmos and the laboratory," Soviet Physics, Uspekhi, vol. 20, pp. 319-334, 1977.

4. L. Grishchuk and A. Polnarev, “Gravitational waves and their interaction with matter and fields,” in General Relativity and Gravitation, One hundred years after the birth of Albert Einstein (A. Held, ed.), vol. 2 (new York), pp. 393-434, Plenum Press, 1980.

5. R. Pound and G. Rebka, “Apparent weight of photons,” Physical Review Letters, vol. 4, pp. 337-341, 1960.

6. J. Mizuno, A. Rüdiger, R. Schilling, W. Winkler, and K. Danzmann, “Frequency response of Michelson- and Sagnac-based interferometers,” Optics Communications, vol. 138, pp. 383-393, 1997.