Fusion Chain Reaction

By M. Gryziński *

CHAIN REACTION WITH CHARGED PARTICLES

With the discovery of the fission chain reaction with neutrons, the possibility of obtaining a chain reaction with charged particles was abandoned because of the small efficiency of charged particles in nuclear reactions. In the most advantageous case, T (D, n) He^4, the efficiency attained is only $\sim 5 \times 10^{-3}$ reactions per 14 Mev deuteron. However, no note was taken of the fact that the efficiency depends on the physical conditions and in some cases may be greatly increased. This is especially true for nuclei of small charge, where the factor of Coulomb barrier penetration is not too high. The development of the chain reaction with charged particles is therefore possible only for light nuclei, where the release of nuclear energy is due to the process of fusion. Only highly exoenergetic reactions of large cross sections may lead to the fusion chain reaction; these are the same reactions as those which are involved in thermonuclear reactions.[1]

FORMULATION OF THE PROBLEM

The mechanism of a fusion chain reaction, which is due to *in statu nascendi* reactions, is as follows. In an exoergic reaction A + B, in which weakly bound groups of nucleons of nuclei A and B form strongly bound groups of reaction products, we obtain particles having kinetic energy. Part of their kinetic energy is transferred in elastic collisions directly to the A and B nuclei of the medium. The recoiling A and B nuclei, in the process of slowing down to thermal energy, have some probability of leading again to the reaction A + B. Under normal physical conditions the dissipation of the energy of the charged particles in collisions with electrons is so large that their range, L, in the medium is much smaller than the mean-free path, λ, with respect to the nuclear process. Therefore, only a small fraction of recoil nuclei lead again to the A + B reaction. The development of an avalanche is possible when the sum of ranges of the recoil nuclei $\sum_i L_i$ is comparable to λ. Since $\sum_i L_i \simeq E_Q(dE/dx)^{-1}$ and $\lambda \simeq (N\langle\sigma\rangle)^{-1}$, where E_Q denotes the kinetic energy released in the A + B reaction, dE/dx the average energy losses of recoil nuclei per unit path length, N the density of reacting nuclei of the medium, $\langle\sigma\rangle$ the mean cross section for the A + B reaction, we can write

$$NE_Q\langle\sigma\rangle/(dE/dx) \sim 1. \tag{1}$$

If we assume $\langle\sigma\rangle \sim 10^{-24}$ cm^2, $E_Q \sim 10$ Mev, we find that the atomic stopping power or stopping factor $\langle dE/dx\rangle/N = \langle\sigma\rangle\, E_Q$ would be 10^{-17} ev atom^{-1} cm^2. Under normal physical conditions it is about a thousand times higher,[2, 3] and we are far from satisfying the criterion (1).

The main idea of the problem involves the dependence of the atomic stopping factor on the physical conditions.

Under normal physical conditions, in the moderate energy range, the most important losses of energy of heavy charged particles are due to scattering from electrons. They are about four thousand times higher than the energy losses in all other processes.[3] The atomic stopping factor due to scattering from electrons was discussed in detail by the author,[4] and, according to Eq. (6) of Ref.4, for a particle ξ having a velocity V_ξ and a charge $Z_\xi e$, it is

$$\sigma\left(\frac{dE}{dx}\right)_{\mathrm{el}}^{\mathrm{sc}} = \frac{4\pi e^4}{mV_\xi^2} Z_\xi^2 \int_0^\infty f(V_e);\, G[(V_e)]dV_e \tag{2}$$

where $f(V_e)$ is the momentum distribution of electrons in the medium and G the universal stopping power function given by Eq. (8) of Ref. 4. There it was shown that these losses depend mainly on the velocity distribution of the electrons, especially in the case $V_\xi \leq V_e$. In the limiting case $V_\xi \ll V_e$, the asymptotic value of G becomes $\frac{2}{3}(V_\xi/V_e)^3$ whereupon we have

$$\sigma\left(\frac{dE}{dx}\right)_{\mathrm{el}}^{\mathrm{sc}} \sim \frac{1}{Ve^3}. \tag{3}$$

Hence we see that the energy losses connected with the scattering from electrons decrease very strongly with their velocity. The electron momentum distribution can be shifted into higher velocities by a considerable rise of temperature or by increasing the density up to the strong degeneracy of the electron gas. In this way we can decrease the atomic stopping factor so that the condition (1) is fulfilled.

Atomic Stopping Factor

As mentioned above, the main energy losses of charged particles are due to scattering from electrons,

* Institute of Experimental Physics, Warsaw University, Warsaw, Poland.

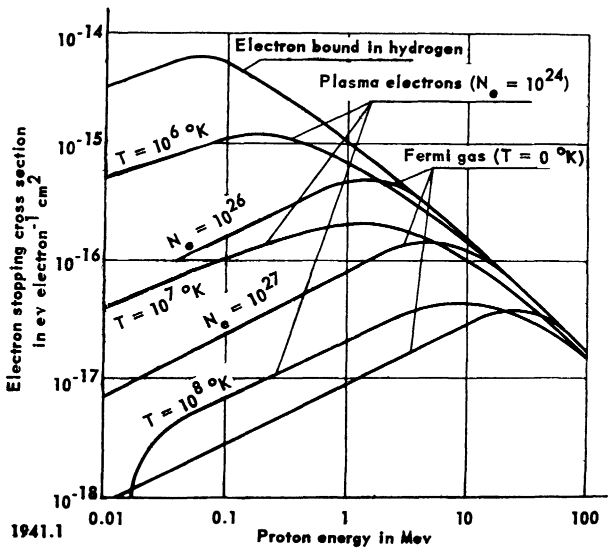

Figure 1. Theoretical calculations of the stopping power of electrons bound in hydrogen, electrons of plasma at different temperatures, and electrons of a Fermi gas

which scattering depends on the state of the medium.

To evaluate the stopping factor of plasma electrons [5] we have to use the Maxwellian momentum distribution in Eq. (2). We obtain an approximate dependence on the temperature of the plasma if we make the substitution $f(V_e) = \delta\ (\langle V_e \rangle - V_e)$, where $\langle V_e \rangle = (8kT/\pi m)^{\frac{1}{2}}$ is the mean thermal velocity of electrons. In the case of interest, $V_\xi \ll V_e$, we have

$$\sigma\left(\frac{dE}{dx}\right)^{\mathrm{sc}}_{\text{plasma electrons}} \simeq -\frac{8\pi e^4}{3m}\left(\frac{\pi m}{8kT}\right)^{3/2} Z_\xi^2\, V_\xi. \quad (4)$$

Similarly, taking into account the momentum distribution of a Fermi gas, we obtain the stopping factor of Fermi gas electrons (*cf.* Eq. (18), Ref. 4; also Ref. 6)

$$\sigma\left(\frac{dE}{dx}\right)^{\mathrm{sc}}_{\text{Fermi electrons}} \simeq -\frac{1}{N_e}\left(\frac{4}{3\pi}\right) \times \left(\frac{2e^4m^2}{\pi^3}\right) Z_\xi^2 V_\xi \ln\left(\frac{V_{\max}\pi}{e^2}\right) \quad (5)$$

where $V_{\max} = (3\pi^2)^{\frac{1}{3}}(\hbar/m)N_e^{\frac{1}{3}}$, N_e the number of electrons per cm^3.

The results of exact computations, where for the maximum impact parameter we have put $D_{\max} = N_e^{-\frac{1}{3}}$ (Appendix 1), are plotted in Fig. 1 for various temperatures and densities.

A decrease in the electron scattering losses increases the role of energy losses connected with the interaction with the nuclei of a medium.

The contribution to the atomic stopping factor due to elastic scattering from nuclei of mass m_{A} and charge $Z_{\mathrm{A}}e$ is:

$$\sigma\left(\frac{dE}{dx}\right)^{\text{elastic sc}}_{\mathrm{nuc}} \simeq -\frac{4\pi e^4}{m_{\mathrm{A}} V_\xi^2}(Z_\xi Z_{\mathrm{A}})^2 \ln\left(\frac{\mu_{\xi\mathrm{A}}\, V_\xi^2}{Z_\xi Z_{\mathrm{A}}\, e^2 N^{\frac{1}{3}}}\right) - \tfrac{1}{2} K_{\xi\mathrm{A}} E_\xi\, \sigma_{\xi\mathrm{A}}{}^{\mathrm{sc}}\ (\sigma_\xi) \quad (6)$$

where $\mu_{\xi\mathrm{A}}$ is the reduced mass, $K_{\xi\mathrm{A}} = 4m_\xi m_{\mathrm{A}}/(m_\xi + m_{\mathrm{A}})^2$ and $\sigma_{\xi\mathrm{A}}{}^{\mathrm{sc}}$ the elastic nuclear scattering cross section of particle ξ from the nuclei A, and N the number of nuclei per cm^3. The first term in Eq. (6) represents the Coulomb scattering, and the second the nuclear scattering, which we assumed isotropic in the center-of-mass system.

The stopping factor due to inelastic collisions with nuclei is:

$$\sigma\left(\frac{dE}{dx}\right)^{\text{inel sc}}_{\mathrm{nuc}} = \Sigma_i \Delta E_i \sigma_{\xi\mathrm{A}}{}^{\text{inel sc}}(E_\xi). \quad (7)$$

The sum is taken over all channels with the excitation energies ΔE_i.

The energy losses connected with the bremsstrahlung [7] of heavy charged particles are very low in comparison with the losses given above; therefore, they can be safely neglected.

Finally, the stopping factor of nucleus A and its Z_{A} electrons is:

$$\sigma\left(\frac{dE}{dx}\right) = Z_{\mathrm{A}}\sigma\left(\frac{dE}{dx}\right)^{\mathrm{sc}}_{\mathrm{el}} + \sigma\left(\frac{dE}{dx}\right)^{\text{el sc}}_{\mathrm{nuc}} + \sigma\left(\frac{dE}{dx}\right)^{\text{inel sc}}_{\mathrm{nuc}}. \quad (8)$$

The total atomic stopping factor of hydrogen plasma† for protons and the relative contribution of their components in various conditions are plotted in Fig. 2.

Evaluation of Multiplication Factor

To determine the exact conditions for the development of an avalanche, we shall examine an infinite homogeneous medium formed by a mixture of two

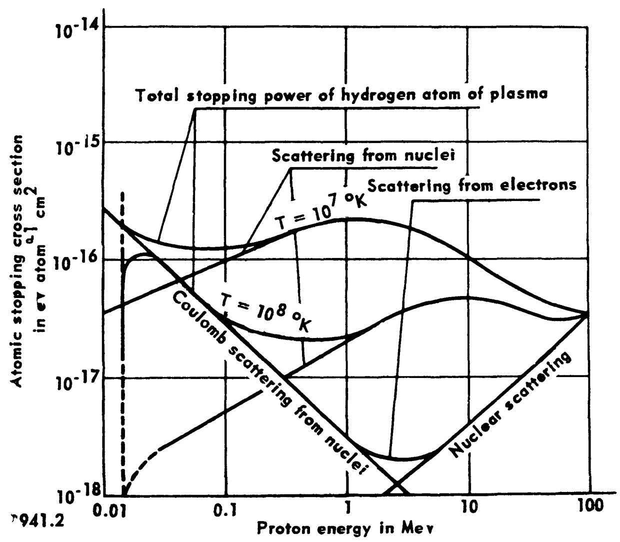

Figure 2. The stopping power of hydrogen plasma and the relative contribution of its components

† As shown above, the stopping factor of hydrogen plasma under the conditions existing in the sun, $\sim 2 \times 10^7$ °K, is about one hundred times lower than that of hydrogen under normal physical conditions. Therefore, Bethe's calculations [8] of the efficiency of reactions *in statu nascendi* in the sun (with the assumption that the energy losses are approximatively the same in both cases) are not valid.

kinds of nuclei, A and B, which can initiate the exoergic reaction. We denote by N_A and N_B the number densities of reacting particles, by $\sigma_{AB}{}^{\xi}$ the laboratory cross section for the reaction A + B (the bombarding particle is denoted by the first lower index) with the emission of the particle ξ. The particles, of high kinetic energy, obtained from this reaction produce a certain number of recoil nuclei.

If $f_{\xi}(E_{\xi}{}^0)$ is the energy distribution of the ξ particles obtained from each reaction A + B, then the number of ξ particles in the energy interval, $E_{\xi}{}^0$ to $E_{\xi}{}^0 + dE_{\xi}{}^0$, is $f_{\xi}(E_{\xi}{}^0)dE_{\xi}{}^0$. Since the major part of the reaction A + B in the avalanche occurs in the moderate energy range (100 — 500 kev) and since the reaction A + B is strongly exoenergetic, we have assumed that this distribution is independent of the energy of the entrance channel. If in the result of reaction A + B we obtain two particles, the function $f_{\xi}(E_{\xi}{}^0)$ is the $\delta(E_{\xi}{}^0 - E_{\xi})$ function. Owing to the destruction of particles ξ on interaction with the A and B nuclei, the initial number $f_{\xi}(E_{\xi}{}^0)dE_{\xi}{}^0$ of particles along the path x drops to the value $q\ (E_{\xi}{}^0,x)f_{\xi}(E_{\xi}{}^0)dE_{\xi}$ where

$$q\left(E_{\xi}{}^0,\ x\right) = \exp - \int_0^x \left(N_A\ \sigma_{\xi A} + N_B\ \sigma_{\xi B}\right) dx \quad (9)$$

and $\sigma_{\xi A}(\sigma_{\xi B})$ is the total reaction cross section of the particle ξ with the nucleus A(B). Taking into account that the energy of particle ξ on the path x drops from $E_{\xi}{}^0$ to E_{ξ} because of energy losses, we can write the last expression in terms of E_{ξ}

$$q(E_{\xi}{}^0, E_{\xi}) = \exp - \int_{E_{\xi}{}^0}^{E_{\xi}} \frac{N_A\,\sigma_{\xi A} + N_B\,\sigma_{\xi B}}{(dE_{\xi}/dx)}\, dE_{\xi} \quad (10)$$

where (dE_{ξ}/dx) denotes the loss of energy of the particle per unit path length. Introduce

$$\sigma_{\xi A}{}^{sc}\ (E_{\xi},\ E_A)\ dE_A,$$

the cross section for the production of recoil nuclei A of energy E_A to $E_A + dE_A$ by the particle ξ with energy E_{ξ}. Then the number of the recoil A nuclei, with energy interval E_A to $E_A + dE_A$ produced by the particles ξ from the reaction A + B along their paths, is

$$g_{\xi A}(E_A)\,dE_A = dE_A \int f_{\xi}(E_{\xi}{}^0)dE_{\xi}{}^0 \int_{E_{\xi}{}^0}^{E_A/K_{A\xi}} N_A \sigma_{\xi A}{}^{sc} \times (E_{\xi}, E_A) q(E_{\xi}{}^0, E_{\xi}) \frac{dE_{\xi}}{(dE_{\xi}/dx)} \quad (11)$$

$\sigma_{\xi A}{}^{sc}(E_{\xi}, E_A)$ is given by the scattering differential cross section $\sigma_{\xi}{}^{sc}(E_{\xi},\ \theta)$ and the relation between the angle of scattering and loss of energy in the collision.[9]

Summing up all the products of the A + B reaction we obtain:

$$g_A{}^{(1)}(E_A) = \textstyle\sum_{\xi} g_{\xi A}(E_A). \quad (12)$$

We can write a similar expression for the energy distribution of recoil nuclei B. As a result of the elastic scattering of the first generation of A and B nuclei we obtain the second generation of recoil nuclei of the medium. With the help of the above considerations we can write the energy distribution for the nth generation of recoil nuclei A:

$$g_A{}^{(n)}(E_A) = \int g_A{}^{(n-1)}(E'_A)dE'_A \int_{E'_A}^{E_A} N_A\,\sigma_{AA}{}^{sc}(E''_A, E_A) \times q(E'_A, E''_A) \frac{dE''_A}{(dE''_A/dx)} + \int g_B{}^{(n-1)}(E'_B) \times dE'_B \int_{E'_B}^{E_A/k_{AB}} N_B\,\sigma_{BA}{}^{sc}(E''_B E_A) q(E'_B, E''_B) \frac{d''E_B}{(dE''_B/dx)}. \quad (13)$$

If we add the energy distributions of all generations we obtain the energy distribution of the whole cascade initiated by the particles from the reaction A + B:

$$G_A(E_A) = \textstyle\sum_n g_A{}^{(n)}(E_A). \quad (14)$$

Having obtained the distributions $G_A(E_A)$ and, in a similar way, $G_B(E_B)$, we can give the number of A + B reactions in the slowing-down process of the cascade initiated by the particles from the one reaction A + B:

$$k = \int G_A(E_A)dE_A \int_{E_A}^{0} N_B\,\sigma_{AB}(E'_A) q \times (E_A, E'_A) \frac{dE'_A}{(dE'_A/dx)} + \int G_B(E_B)dE_B \int_{E_B}^{0} N_A\,\sigma_{BA}(E'_B) q \times (E_B, E'_B) \frac{dE'_B}{(dE'_B/dx)}. \quad (15)$$

If the reaction A+B has only one exoergic channel the number k is the multiplication factor for the given medium. The condition for the development of the avalanche, therefore, is $k > 1$.

In the numerical calculations, as long as the slowing down process of the products of reaction A + B and recoil nuclei is due to scattering from electrons, we can take into consideration only the first generation of recoil nuclei. Then from Eqs. (4), (5) and (15) we have:

a. in the case of charged products of the reaction A+B

$$k \sim \left[\sigma\left(\frac{dE}{dx}\right)_{el}^{sc}\right]^{-2} \sim \begin{cases} T^3 & \text{for plasma} \\ N_e{}^2 & \text{for degenerate medium} \end{cases} \quad (16)$$

b. in the case of neutrons

$$k \sim \left[\sigma\left(\frac{dE}{dx}\right)_{el}^{sc}\right]^{-1} \sim \begin{cases} T^{3/2} & \text{for plasma} \\ N_e & \text{for degenerate medium} \end{cases} \quad (17)$$

Critical Mass

All our present considerations concern the conditions for the development of fusion chain reactions in infinite media. In a finite medium the conditions are different and a critical mass exists as in a fission chain reaction.

In the first approximation, the critical mass can be estimated very easily if we consider that the mean-free path λ with respect to the elastic scattering of the particles taking part in the reaction has to be comparable with the dimension L of the system. If we denote by N the number of nuclei of a medium in a

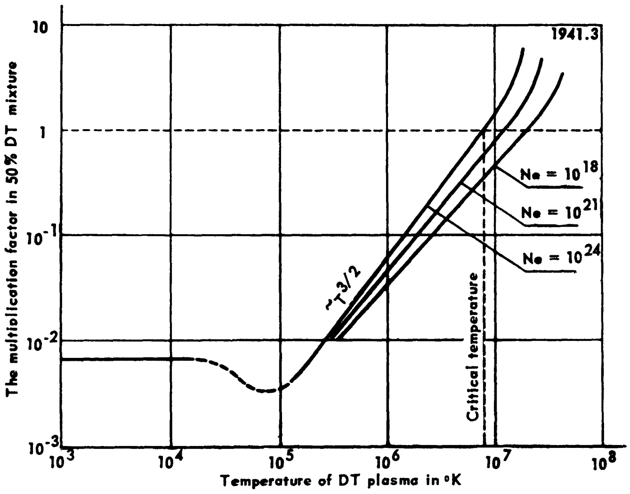

Figure 3. The multiplication factor in 50% D-T plasma as a function of temperature

unit volume, by m the mass of a nucleus in grams, and by $\langle\sigma\rangle^{sc}$ the cross section for elasting scattering, we obtain $m_{cr} \approx m_1 N_A{}^3 = m_1(\langle\sigma\rangle^{sc})^{-3} N^{-2}$. Taking into account that $\langle\sigma\rangle^{sc} \approx 10^{-24}$ cm^2 and $m_1 \approx 10^{-24}$ g, the critical mass in grams is

$$m_{cr} \approx 10^{48}/N^2. \tag{18}$$

We see that the critical mass is very strongly dependent on the density of a medium. For densities $N = 10^{28}$, 10^{24}, 10^{20} nuclei/cm^3, the critical masses are 10^{-8}, 10^0, 10^8 grams respectively.

NUMERICAL CALCULATIONS

Now, to illustrate the theory given above we shall determine the conditions for the development of a fusion chain reaction in a D-T mixture.

As a result of reaction D + T we obtain alphas and neutrons with energies $\sim$ 3.5 Mev and $\sim$ 14.1 Mev respectively. Because of the much greater initial energy and much lower energy losses for the neutrons, most recoil nuclei D and T result from scattering of neutrons; therefore, according to Eq. (12), $g_D{}^{(1)}(E_D) \approx g_{nD}(E_D)$ and $g_T{}^{(1)}(E_T) \approx g_{nT}(E_T)$. Taking into account the fact that the absorption of fast and intermediate neutrons in the D-T medium is negligibly small we have $q_n \approx 1$. Assuming the scattering of neutrons from D and T nuclei isotropic in the center-of-mass system, we obtain

$$\sigma_{nD}{}^{sc}(E_n, E_D) \approx \frac{\sigma_{nD}{}^{sc}(E_n)}{K_{nD}E_n},$$

$$\sigma_{nT}{}^{sc}(E_n, E_T) \approx \frac{\sigma_{nT}{}^{sc}(E_n)}{K_{nT}E_n}. \tag{19}$$

Because, in the first approximation, the slowing down of neutrons is due to elastic scattering from D and T nuclei

$$dE_n/dx \approx \tfrac{1}{2} K_{nD}E_n\, \sigma_{nD}{}^{sc}N_D + \tfrac{1}{2} K_{nT}E_n\, \sigma_{nT}{}^{sc}N_T \tag{20}$$

the energy distributions of the first generation of recoil nuclei are, respectively:

$$g_{nD}(E_D) \approx \int_{14.1}^{E_D/k_{nD}} \frac{dE_n}{(k_{nD}E_n)^2} \times \frac{1}{1 + \sigma_{nT}{}^{sc}K_{nT}N_T/\sigma_{nD}{}^{sc}K_{nD}N_D} \tag{21a}$$

$$g_{nT}(E_T) \approx \int_{14.1}^{E_T/k_{nT}} \frac{dE_n}{(K_{nT}E_n)^2} \times \frac{1}{1 + \sigma_{nD}{}^{sc}K_{nD}N_D/\sigma_{nT}{}^{sc}K_{nT}N_T}. \tag{21b}$$

Since the energy losses of deuterons and tritons up to 10^7 °K, in the case of a plasma, and up to 10^3 g/cc, in the case of a degenerate medium, are due to scattering from electrons we can write

$$k \approx \frac{N_D}{N_D+N_T} \int g_{nD}(E_D)dE_D \int_{E_D}^{0} \frac{\sigma_{DT}}{\sigma\left(\frac{dE'_D}{dx}\right)_{el}^{sc}} dE'_D + \frac{N_T}{N_D+N_T} \int g_{nT}(E_T)dE_T \int \frac{\sigma_{TD}}{\sigma\left(\frac{dE'_T}{dx}\right)_{el}^{sd}} dE'_T \tag{22}$$

where $\sigma\left(\frac{dE_D}{dx}\right)_{el}^{sc}$ and $\sigma\left(\frac{dE_T}{dx}\right)_{el}^{sc}$ are given by Eq. (2).

The value of the multiplication factor obtained by the numerical calculations for a 50% D-T mixture under various conditions are plotted in Figs. 3 and 4. The cross sections we have put equal to the geometrical cross section and σ_{DT} is taken from Bame and Perry.[10]

CONCLUSIONS

The role of the *in statu nascendi* reactions in the release of nuclear energy depends on the physical conditions and, in the case of high temperature or high density, they are decisive. If we denote by E_{th} the energy released in a unit volume in the thermo-

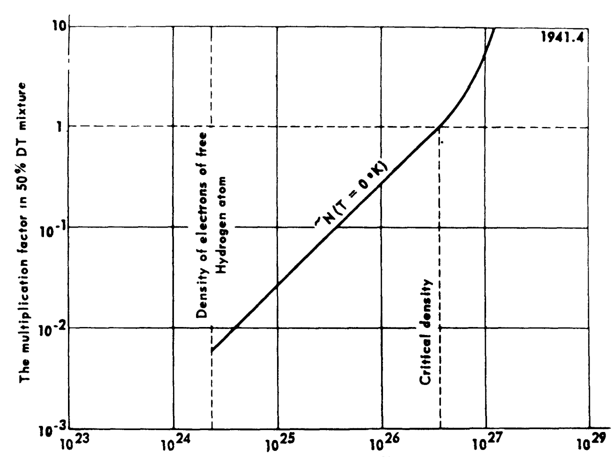

Figure 4. The multiplication factor in 50% D-T medium as a function of density for T = 0°K

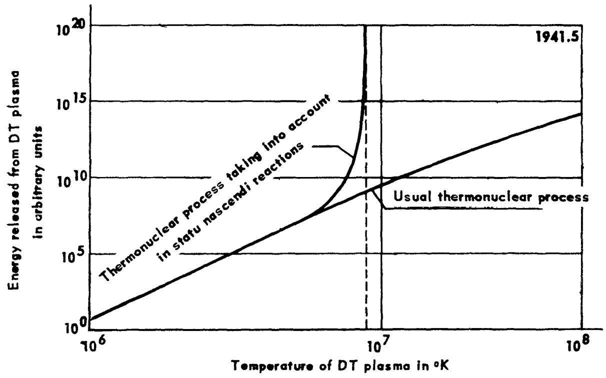

Figure 5. The release of energy in exoergic mixture for the usual thermonuclear process, and for a process taking into account *in statu nascendi* reactions

nuclear process, then the energy released in a unit volume with respect to the *in statu nascendi* reactions is

$$E_{\mathrm{tot}} = E_{\mathrm{th}} \,/\, (1 - k(T, N)),$$

where k is the multiplication factor for the given medium. The factor k depends on the temperature of the medium—or, more accurately, on the temperature of its electrons—and on its density. As the multiplication factor approaches unity, the process of energy release has an avalanche character and the entire nuclear energy of an exoergic mixture is released instantaneously. The stationary state for a slow release of energy does not exist above the critical temperature or above the critical density; even at a temperature of absolute zero the exoergic mixture is explosive.

A plot of E_{th} (Ref. 11) and E_{tot} as a function of temperature for 50% D-T mixture is given in Fig. 5.

APPENDIX

As was pointed out previously,[4] the maximum impact parameter is, in general, a function of the velocities of interacting particles, their masses and charges, as well as of the external fields. In each problem this parameter must be determined separately.

In the case of electrons bound in atoms, or Fermi gas electrons, the determination of the maximum impact parameter does not present any difficulty, but in the case of plasma electrons it is the subject of many discussions. According to Cowling,[12] Chandrasekhar [13] and others, it is suitable to put the maximum impact parameter equal to the mean distance between the ions, but according to Landau,[14] Cohen, Spitzer and Routly [15] and others, it must be equal to the Debye radius.

From Eq. (2) it follows at once that in the limiting case $V_\xi \ll V_\mathrm{e}$ the atomic stopping cross section is independent of the assumed value of $D_{\max}$.

To determine the maximum impact parameter in the second limiting case, $V_\mathrm{e} \ll V_\xi$, we must take into account the fact that the charged particles of the plasma are interacting with each other. Consider two particles with charges $+Ze$ and $-Ze$ and the distance r between them: the Coulomb force between them is $(Ze/r)^2$. The transfer of momentum to such a binary system from a particle ξ is negligibly small when the force of interaction between the particle ξ and each particle of the system is less than the force of internal interaction, or: $Z^2(e/r)^2 \gtrsim ZZ_\xi(e/D)^2$. Taking into account the mean value of distance between charged particles in the plasma we finally obtain $D_{\max} \approx N^{-\frac{1}{3}}$.

The assumption that the maximum impact parameter is equal to the Debye radius will slightly change the numerical results owing to the logarithmic dependence on $D_{\max}$.

REFERENCES

1. W. B. Thompson, Proc. Phys. Soc., *70*, 1 (1957).
2. P. K. Weyl, Phys. Rev., *91*, 289 (1953).
3. S.K. Allison and S. D. Warshaw, Revs. Modern Phys.. *25*, 779 (1953).
4. M. Gryziński, Phys. Rev., *107*, 1471 (1957).
5. E. N. Parker, Phys. Rev., *10*, 830 (1957).
6. E. Fermi and E. Teller, Phys. Rev., *72*, 399 (1957).
7. L. Landau and L. Lifshitz, *Field Theory*, pp. 208, 219, sec. ed., Moscow-Leningrad (1948).
8. H. Bethe, Phys. Rev., *55*, 434 (1938).
9. E. Segre, *Experimental Nuclear Physics*, pp. 9, 14, Vol. II, first ed., New York, London (1953).
10. S. J. Bame and J. Perry, Phys. Rev., *107*, 1616 (1957).
11. R. F. Post, Revs. Modern Phys., *28*, 338 (1956).
12. T. G. Cowling, Proc. Roy. Soc., *A138*, 453 (1945).
13. S. Chandrasekhar, Astrophys. J., *97*, 255, 263 (1943).
14. L. D. Landau, Sow. Phys., *10*, 154 (1936).
15. R. S. Cohen, L. Spitzer and P. M. Routly, Phys. Rev., *80*, 230 (1950).