Exact Revival of the Bound Wave Function of Hydrogen for Arbitrary Quantum State

Matt Kalinski
Department of Chemistry and Biochemistry
Utah State University, Logan UT 84322-0300
mkal@cc.usu.edu

Abstract: We show that for the hydrogen atom for the state consisting of arbitrary but final number of the bound states total and exact full revival of the wavefunction exists for sufficiently long time of evolution.

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It has been found out that quantum systems may exhibit sudden recovery of the dynamics after very long time of apparent decay of their expectation values and growth of their fluctuations. This can be two level atom interacting with the quantum modes of the electromagnetic field [1] or the hydrogen atom [2]. Further the wave function not only can recover its initial condition but also split itself in self similar parts during this recovery in fractional revivals. This recovery was pointed out approximate for the previously discussed quantum systems other then the harmonic oscillator with coherent state evolution.

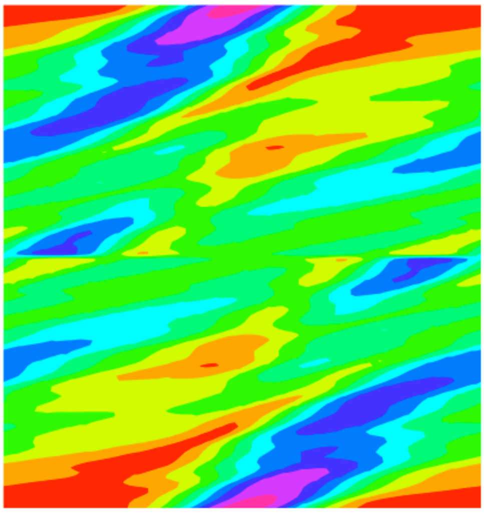

Fig. 1 The simplified wavefunction for the function (1) defined as $g(\phi,t) = (\sum_{n_0}^{n_f} e^{-(n-n_0)^2/4}\times e^{-iE_nt/\hbar}e^{i(n-n_0)\phi}$ plotted (modulus) as the function of time (vertical) and the angle just after $t=0$ and before recovery from Gaussian to Gaussian exactly ($n=1-10$ and $n_0=5$).

The energies of hydrogen atom within the nonrelativistic Schrödinger equation approximation are given by

$$E_n = -\frac{\mathcal{R}_\infty}{n^2} \tag{1}$$

The quantum wave function of Hydrogen spanned by the arbitrary large but finite number of eigenstates ($\mathcal{R}_\infty = 13.6eV$) at the given time t is

$$\Psi(t) = \sum_{n_0}^{n_f} c_n \phi_n e^{-iE_nt/\hbar} \tag{2}$$

The necessary and sufficient condition for the total revival of the wave function is

$$e^{iE_n(t+T)/\hbar} = e^{iE_nt/\hbar} \tag{3}$$

i.e when all phases will made the multiple complex circle evolution or

$$E_n T/\hbar = 2\pi N_n \tag{4}$$

for each quantum number n, N_n is an integer different for each n. Remarkably for the hydrogen atom as for the harmonic oscillator this condition can be exactly fulfilled even if the spectrum is highly nonlinear. To have this condition fulfilled it is sufficient to assume that the harmonic oscillator exists for which the energies are those of the hydrogen atom but leaving an arbitrary and necessary holes in the spectrum i.e. there is ω and n_i such that for each i the hydrogen energy E_i is

$$E_i = \omega n_i \tag{5}$$

Substituting (1) we get

$$\frac{\mathcal{R}_\infty}{i^2} = \omega n_i \tag{6}$$

or

$$\mathcal{R}_\infty/\omega = const = integer = i^2 n_i \tag{7}$$

for each of the contributing state i.

Let $L_M = \mathcal{R}_\infty/\omega$ to be the lowest common multiple of squares n^2 of all quantum numbers of states appearing in the expansion (1) so (7) holds for each i. We therefore find the time after which the wave function recovers exactly to the initial condition i.e. $\Psi(t) = \Psi(t+T)$ (Fig. 1) as

$$T = 2\pi\hbar L_M/\mathcal{R}_\infty \tag{8}$$

while the L_M is found.

Therefore the problem of finding the time of the exact revival for the arbitrary bound quantum state of hydrogen simplifies to finding the lowest common multiple of the squares of quantum number of contributing states. One may notice that this time is much longer than the normal revival time $T_{cl}n_0/3$ [2] (Fig. 1) for energetically dispersed states around n_0 since this number is very large comparing to n_0.

After many attempts of the full normal revival which is never exact and proceeded with the fractional revivals the state will finally evolve to the exact original wave function. Please note that the condition (7) predicts the shortest full revival time since the common multiplicity of i^2 does not need to be he lowest. Fig. 1 shows the space and time evolution of the Gaussian control function constructed from 10 lowest states of hydrogen as it was a simple quantum rotor but with energies of Hydrogen just after $t = 0$ and before the exact revival time showing the exact wave function recovery. The essence to predict it for very extended states is therefore to know the arbitrary large prime number.

For the Trojan wavepacket [3] spanned by 120 states around the central value $n_0 = 60$ according to our Mathieu theory this number is for example 91372196858003047744908687587514511895642264481092498627782681 29174966037136419033787161284104217600000 $\approx 10^{100}$ and the shortest exact revival time is $10^{95}T_{rev}$(approximate full revival time)= $10^{97}T_{cl}$(Kepler period) =$10^{86}s = 10^{79}$ years [2].

References

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