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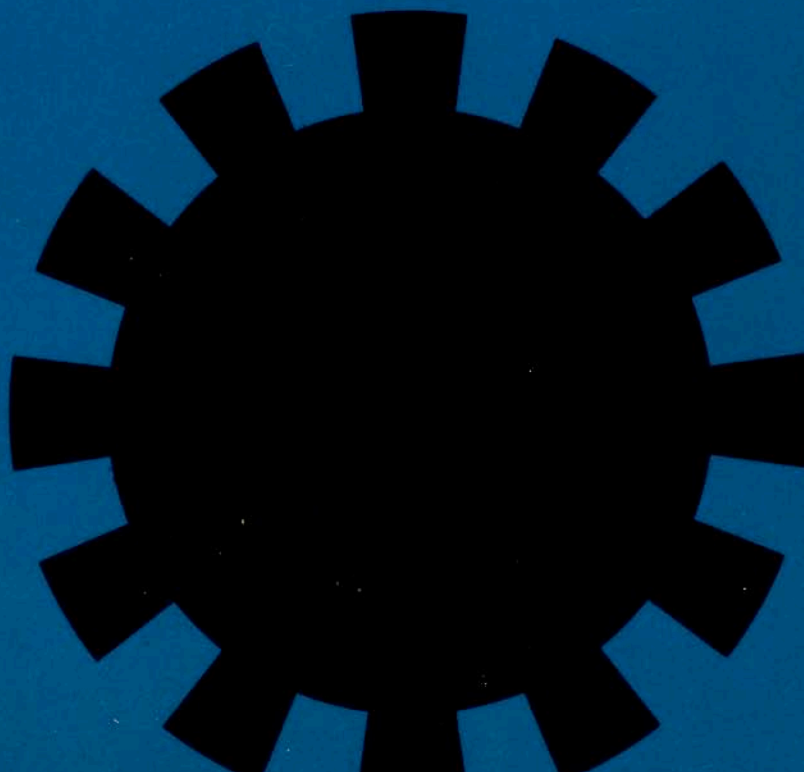
quantum mechanics

THIRD EDITION

INTERNATIONAL STUDENT EDITION



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The position probability density is then

$$|\psi(x,t)|^2 = \left\{2\pi\left[(\Delta x)^2 + \frac{\hbar^2t^2}{4m^2(\Delta x)^2}\right]\right\}^{-\frac{1}{2}} \exp - \frac{x^2}{2[(\Delta x)^2 + \hbar^2t^2/4m^2(\Delta x)^2]} \tag{12.21}$$

Equation (12.21) is of the same form as $|\psi(x,0)|^2$, except that $(\Delta x)^2$ is replaced by $(\Delta x)^2 + \hbar^2t^2/4m^2(\Delta x)^2$, which is equal to $(\Delta x)^2 + (\Delta p)^2t^2/m^2$. Thus the center of the packet remains at $x = 0$ while the breadth of the packet increases as t departs from zero in both past and future directions. The smaller the initial uncertainty in position, the larger the uncertainty in momentum and the more rapidly the packet spreads; the time-dependent part of the above expression, $t(\Delta p)/m$, is simply the distance traveled by a classical particle of momentum Δp in the time t.

Use of the δ-function normalization does not alter the results of the foregoing calculation. The expression for A_k given in Eq. (12.18) is to be multiplied by $(L/2\pi)^{\frac{1}{2}}$; in Eq. (12.19) the summation is to be replaced directly by $\int dk$, thus eliminating a factor $L/2\pi$; finally, u_k in Eq. (12.19) is to be multiplied by $(L/2\pi)^{\frac{1}{2}}$. These three factors cancel, and so Eqs. (12.20) and (12.21) are unaffected by the choice of normalization of the momentum eigenfunctions.

CLASSICAL LIMIT

We have seen in Sec. 7 that a wave packet always moves like a classical particle insofar as the expectation values of its position and momentum are concerned. However, classical dynamics is useful as a description of the motion only if the spreading of the wave packet can be neglected over times of interest in the particular problem.

As a simple example of the kind of parameter that indicates when the classical limit is realized, we consider a wave packet that corresponds to a classical particle moving in a circular orbit of radius a and period T. We shall assume that this packet is sufficiently well localized so that the potential energy does not vary appreciably over its dimensions. Then the classical theory can provide a useful description of the motion only if a wave packet like that discussed above spreads by an amount that is small in comparison with a during a time that is large in comparison with T. The smallest spread of a packet during a time interval of magnitude t is attained when Δx is chosen to be of order $(\hbar t/m)^{\frac{1}{2}}$. We thus require that $(\hbar t/m)^{\frac{1}{2}} \ll a$ when $t \gg T$. This condition may be simply expressed by saying that the angular momentum $2\pi ma^2/T$ of the particle must be very large in comparison with $\hbar$. Thus for most atomic systems, where the angular momentum of each electron is of order $\hbar$, a wave packet corresponding to a well-localized particle spreads so much in one period that this type of description of the motion is not of physical interest.

PROBLEMS

1. Given three degenerate eigenfunctions that are linearly independent although not necessarily orthogonal, find three linear combinations of them that are orthogonal to each other and are normalized. Are the three new combinations eigenfunctions? If so, are they degenerate?

2. Show that so far as the one-dimensional motion of a particle is concerned, the functions $u_{x'}(x) = \delta(x - x')$ for all real x' constitute a complete orthonormal set and that each of them is an eigenfunction of the position variable x with the eigenvalue x'. Set up the position probability function and compare with that obtained in Sec. 7.

3. If the potential energy $V(x)$ in a one-dimensional problem is a monotonic increasing function of x and independent of the time, show that the functions $u_{V'}(x) = (dV/dx)^{-\frac{1}{2}}_{x=x'}\ \delta(x - x')$ for all real x', where $V' = V(x')$, constitute a complete orthonormal set of eigenfunctions of the potential energy with eigenvalues V'. Find the probability function for the potential energy, and show that it has the properties that would be expected of it.

4. What changes are needed in the discussion of the momentum eigenfunctions given in Sec. 11 if the normalization is carried through in a box of rectangular parallelepiped shape rather than in a box of cubical shape?

5. Find two other representations for the Dirac δ function like that given in Eq. (11.9).

6. Verify each of Eqs. (11.13) involving δ functions.

7. Show that the two Eqs. (11.20) are correct: that the momentum probability function defined in Eqs. (11.19) and (11.17) for a normalized ψ sums or integrates to unity.

8. The expression in brackets in the integrand of Eq. (10.19) enables one to calculate ψ at time t' in terms of ψ at time t. If this expression is called $iG(x',t';x,t)$ in the one-dimensional case, then $\psi(x',t') = i\int G(x',t';x,t)\psi(x,t)\,dx$. Show that for a free particle in one dimension

$$G_0(x',t';x,t) = -i\left[\frac{m}{2\pi i\hbar(t'-t)}\right]^{\frac{1}{2}} e^{im(x'-x)^2/2\hbar(t'-t)}$$

Assume that ψ has the form of the normalized minimum wave packet (12.11) at $t = 0$; use the above result to find ψ and $|\psi|^2$ at another time t'. This G_0 is called the free-particle Green's function in one dimension (see Sec. 36).

9. Let $u_1(\mathbf{r})$ and $u_2(\mathbf{r})$ be two eigenfunctions of the same hamiltonian that correspond to the same energy eigenvalue; they may be the same function, or they may be degenerate. Show that

$$\int u_1^*(\mathbf{r})(xp_x + p_x x)u_2(\mathbf{r})\,d^3r = 0$$

where the momentum operator $p_x = -i\hbar(\partial/\partial x)$ operates on everything to its right. What is the relation of this problem to Prob. 5, Chap. 2?