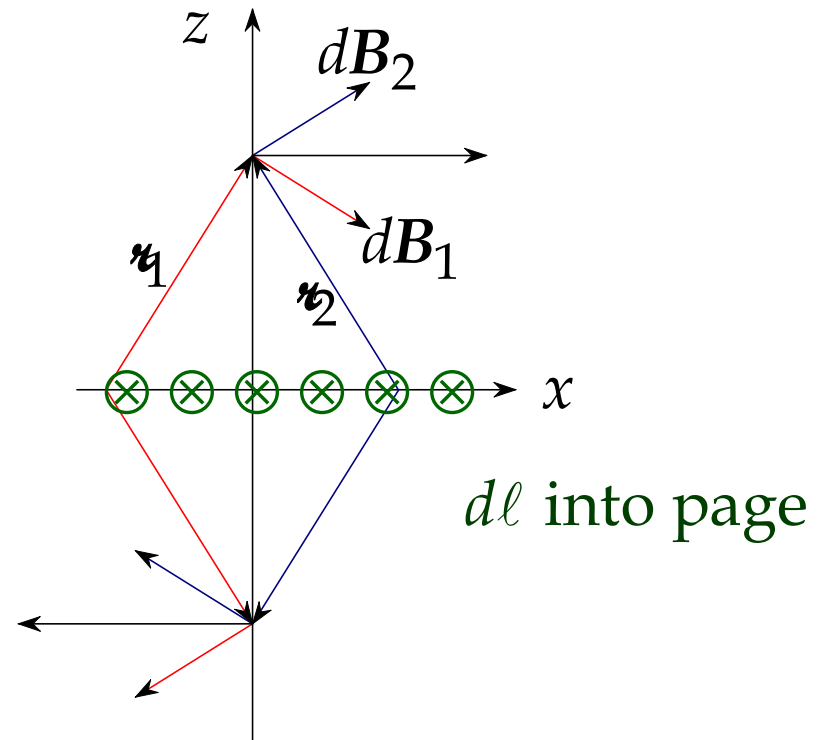

Today in Physics 217: magnetic vector potential

- Potentials, and the magnetic vector potential A
- Arbitrariness of the potential: the Coulomb gauge
- A and the Biot-Savart law
- What is A ?
- Magnetic boundary conditions



Potentials

Recall the Helmholtz theorem (lecture, 16 September): any vector function F can be expressed as

$$F = -\nabla U + \nabla \times W$$

where

$$U(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} D(\mathbf{r}') d\tau'$$

Scalar and vector potential

$$W(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{C}(\mathbf{r}') d\tau'$$

and under the assumptions that, as $r \rightarrow \infty$,

$$F \rightarrow 0, \quad r^2 \nabla \cdot F \rightarrow 0, \quad \text{and} \quad r^2 \nabla \times F \rightarrow 0 \quad .$$

Potentials (continued)

We have also discussed two immediate consequences of the Helmholtz theorem :

□ **Irrotational fields.** If a vector function is such that $F = -\nabla U$, then all of the following are true:

$$\nabla \times F = 0.$$

$\int_a^b F \cdot dl$ is independent of path, given a and b .

$$\oint F \cdot dl = 0 \quad (\text{thus "irrotational"}).$$

In electrostatics, the electric field E is irrotational, and is the gradient of the (scalar) potential V .

Potentials (continued)

- **Solenoidal fields.** If a vector function is such that $F = \nabla \times W$, then all of the following are true:

$$\nabla \cdot F = 0.$$

$\int_S F \cdot da$ is independent of surface, given the boundary C .

$$\oint F \cdot da = 0.$$

In magnetostatics, the magnetic field B is solenoidal ($\nabla \cdot B = 0$), and is the curl of the **magnetic vector potential**:

$$B = \nabla \times A \quad .$$

Potentials (continued)

Recall also that although F is a unique solution to the differential equations, the potentials U and W are not, just because the field depends upon the gradient and curl of these potentials, and:

- if a constant is added to U , the result has the same gradient, because the gradient of a constant is zero.
- Similarly, if a gradient is added to W , the result has the same curl, because the curl of a gradient is zero.

Thus we can add an arbitrary constant to a scalar potential, and the gradient of an arbitrary scalar function to a vector potential, without changing the physics. To make special choices of these “offsets” can make certain problems easier to set up. A set of such choices is called a **gauge**.

Arbitrariness of A : the Coulomb gauge

A is arbitrary in the sense that one can add a gradient to it without changing B , because the curl of any gradient is zero:

$$\nabla \times (\mathbf{A} + \nabla \lambda) = \nabla \times \mathbf{A} + \nabla \times \nabla \lambda = \nabla \times \mathbf{A} \quad .$$

As was the case for V , there is a *conventional* reference point, corresponding to our conventional choice of $V(\mathbf{r}) \rightarrow 0$ as $r \rightarrow \infty$. The convention comes from a consideration of Ampère's

Law:
$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$= \frac{4\pi}{c} \mathbf{J} \quad .$$

If $\nabla \cdot \mathbf{A}$ were zero, then Ampère's law would yield an expression resembling the Poisson equation. In electrodynamics this choice is called the **Coulomb gauge**.

Arbitrariness of A : the Coulomb gauge (continued)

$$\nabla \cdot \mathbf{A} = 0 \quad \Rightarrow \quad \nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} \quad ,$$

which might be handy, and also makes magnetostatics look even more like electrostatics.

- But is there always a scalar function λ such that its gradient can be added to A_0 to make a new vector potential with zero divergence? Yes. Here's why. Let

$$\mathbf{A} = \mathbf{A}_0 + \nabla \lambda \quad ,$$

where $\nabla \lambda$ is chosen such that

$$\nabla \cdot \mathbf{A} = 0 = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda \quad \Rightarrow \quad \nabla^2 \lambda = -\nabla \cdot \mathbf{A}_0$$

Arbitrariness of A : the Coulomb gauge (continued)

- But this is just a Poisson equation, and we know it has a solution, so the appropriate function λ exists. We even know what it is, because it is governed by the same relation as the electrostatic scalar potential, for which the general solution is known by Coulomb's law:

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{A}_0(\mathbf{r}')}{r} d\tau' .$$

(That turns out not to be why they call it the Coulomb gauge, though.)

A and the Biot-Savart law

We showed on Wednesday, using the Biot-Savart law, that

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c} \nabla \times \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau' \quad .$$

Apparently,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau' \quad \left(\text{cf. } V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{r} d\tau' \right) \quad .$$

This turns out already to be divergenceless:

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_{\mathcal{V}} \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{r} \right) d\tau' \quad \text{Apply Product Rule \#5:}$$

A and the Biot-Savart law (continued)

$$\begin{aligned}
 \nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{1}{c} \int_{\mathcal{V}} \left[\cancel{\frac{1}{r} \nabla \cdot \mathbf{J}(\mathbf{r}')} + \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{r} \right) \right] d\tau' && \mathbf{J}(\mathbf{r}') \text{ doesn't} \\
 & && \text{depend on } r \\
 &= -\frac{1}{c} \int_{\mathcal{V}} \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{r} \right) d\tau' && \text{because } \nabla \left(\frac{1}{r} \right) = -\nabla' \left(\frac{1}{r} \right) \\
 &= -\frac{1}{c} \int_{\mathcal{V}} \left[\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{r} \right) - \cancel{\frac{1}{r} \nabla' \cdot \mathbf{J}(\mathbf{r}')} \right] d\tau' && \text{magnetostatics!} \\
 &= -\frac{1}{c} \oint_{\mathcal{S}} \frac{\mathbf{J}(\mathbf{r}')}{r} \cdot d\mathbf{a}' && \text{divergence theorem} \\
 &= 0 && \text{by definition, } \mathbf{J} = 0 \text{ everywhere on } \mathcal{S}.
 \end{aligned}$$

What is the magnetic vector potential?

- Unlike V , which we think of as work per unit charge, there's no obvious mechanical interpretation of A .
- Momentum per unit charge (times c) comes closest. For instance, the canonical momentum of a charged particle in an electromagnetic field is

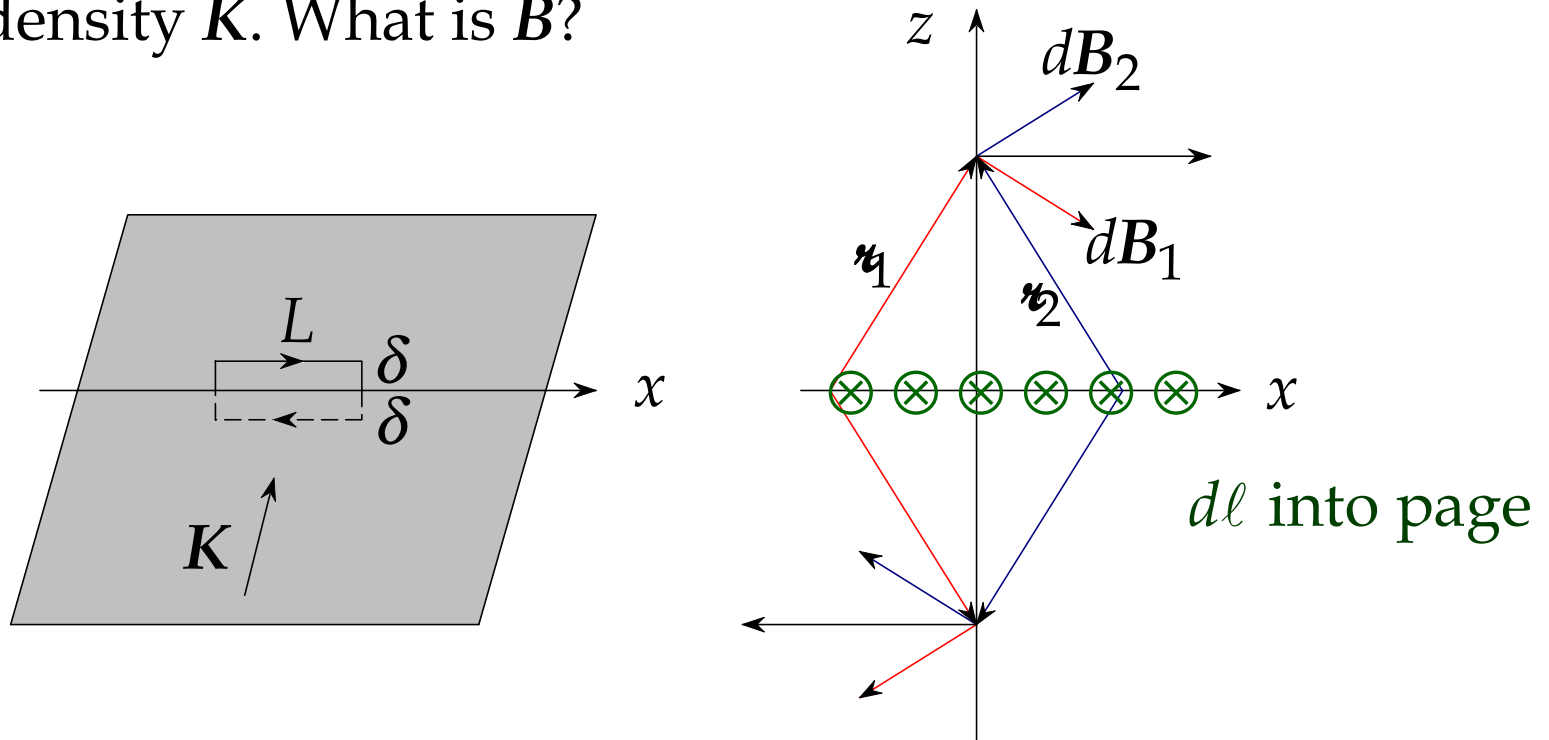
$$\mathbf{p}_{\text{canonical}} = \mathbf{p} - \frac{q}{c} \mathbf{A} \quad ,$$

and as such it appears frequently in the equations of quantum mechanics.

- Also unlike V , it's a vector. Since \mathbf{B} is solenoidal it can't have a scalar potential. Thus A is much less useful in magnetostatic calculations than V is in electrostatics.

Magnetic boundary conditions

Consider an infinite plane covered by a current with constant surface density K . What is B ?



Solve with Ampère's law with the loop as shown, as the vertical components cancel and the horizontal ones add.

Magnetic boundary conditions (continued)

$$\oint \mathbf{B} \cdot d\ell = \frac{4\pi}{c} I_{\text{enclosed}}$$

$$BL + 0 + BL + 0 = \frac{4\pi}{c} KL$$

$$\Rightarrow \mathbf{B} = \frac{2\pi K}{c} \hat{\mathbf{x}} \quad (z > 0), \quad -\frac{2\pi K}{c} \hat{\mathbf{x}} \quad (z < 0).$$

$$\Rightarrow \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \frac{4\pi}{c} \mathbf{K} \times \hat{\mathbf{z}} \quad .$$

This problem is the paradigm for obtaining the boundary conditions for magnetostatic boundary-value problems:

Magnetic boundary conditions (continued)

If $\delta \rightarrow 0$, Ampère's law becomes

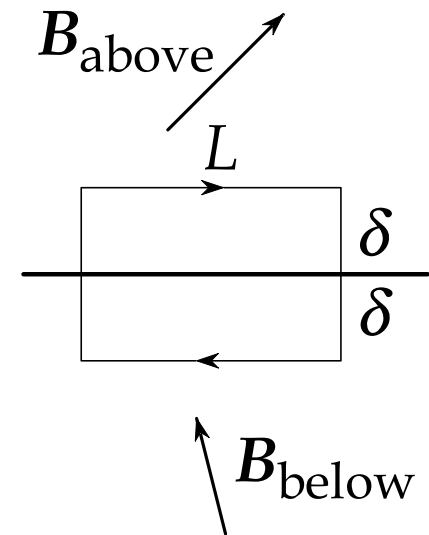
$$\left(B_{\parallel, \text{above}} - B_{\parallel, \text{below}} \right) L = \frac{4\pi}{c} KL \quad ,$$

as the contributions from the perpendicular components of \mathbf{B} vanish as the sides of the loop approach zero size.

Thus

$$B_{\parallel, \text{above}} - B_{\parallel, \text{below}} = \frac{4\pi}{c} K \quad .$$

For the components perpendicular to the surface, use $\nabla \cdot \mathbf{B} = 0$, with a Gaussian pillbox that has very short sides:



Magnetic boundary conditions (continued)

$$\oint B \cdot da = 0$$

$$B_{\perp, \text{ above } A} - B_{\perp, \text{ below } A} =$$

Combine with result for parallel components:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \frac{4\pi}{c} \mathbf{K} \times \hat{\mathbf{n}} \quad .$$

(cf. $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = 4\pi\sigma \hat{\mathbf{n}}.$)

Similarly, $\mathbf{A}_{\text{above}} - \mathbf{A}_{\text{below}} = 0 \quad ,$

$$\frac{\partial}{\partial n} \mathbf{A}_{\text{above}} - \frac{\partial}{\partial n} \mathbf{A}_{\text{below}} = -\frac{4\pi}{c} \mathbf{K} \quad \left(\frac{\partial}{\partial n} \equiv \hat{\mathbf{n}} \cdot \nabla \right).$$

