0.1. **Electrostatics.**

$$\mathbf{F} = \sum_{i=1}^{n} \frac{q q_i}{|\mathbf{r}-\mathbf{r_i}|^3}(\mathbf{r}-\mathbf{r_i})$$

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} \frac{q_i}{|\mathbf{r}-\mathbf{r_i}|^3}(\mathbf{r}-\mathbf{r_i})$$

$$\mathbf{E}(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|^3}(\mathbf{r}-\mathbf{r'})$$

$$\int_S d\mathbf{s} \bullet \mathbf{E}(\mathbf{r}) = 4\pi Q_{enclosed}$$

$$\nabla \bullet \mathbf{E}(\mathbf{r}) = 4\pi\rho(r)$$

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$$

$$\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}$$

$$\Phi(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} d\mathbf{l} \bullet \mathbf{E}$$

$$W = q(\Phi(\mathbf{r_b}) - \Phi(\mathbf{r_a}))$$

$$W_{discrete} = \frac{1}{2}\sum_{i,j,i\neq j} \frac{q_i q_j}{|\mathbf{r_i}-\mathbf{r_j}|}$$

$$W_{continious} = \frac{1}{8\pi}\int d^3r \mathbf{E}^2(\mathbf{r})$$

$$\sigma = \frac{1}{4\pi}\hat{\mathbf{n}} \bullet (\mathbf{E_r} - \mathbf{E_l})$$

0.2. **Multipole.**

$$\Phi(\mathbf{r}) = \sum_{n=1}^{\infty} \frac{qR^n}{r^{n+1}} P_n(cos(\theta)) \quad r >> R$$

$$\Phi(\mathbf{r}) = \sum_{n=1}^{\infty} \frac{qr^n}{R^{n+1}} P_n(cos(\theta)) \quad r << R$$

$$\Phi(\mathbf{r}) = \int \rho(\mathbf{r'}) \sum_{n=0}^{\infty} P_n(cos(\gamma)) \frac{r'^n}{r^{n+1}} d^3r'$$

$$cos(\gamma) = cos(\theta)cos(\theta') + sin(\theta)sin(\theta')cos(\phi-\phi')$$

$$\mathbf{p} = q\mathbf{d} \ or \int d^3r \mathbf{r}\rho(\mathbf{r})$$

$$\Phi_{dipole}(\mathbf{r}) = \frac{\mathbf{p} \bullet \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{dipole}(\mathbf{r}) = \frac{3(\mathbf{p} \bullet \hat{\mathbf{r}}) - \mathbf{p}}{r^3}$$

$$\tau_{dipole} = \mathbf{p} \ x \ \mathbf{E}(\mathbf{r}) + \mathbf{r} \ x \ (\mathbf{p} \bullet \nabla)\mathbf{E}(\mathbf{r})$$

$$Q_{ij} = \int d^3r \ (3x_i xj - \delta_{ij} r^2)\rho(\mathbf{r})$$

$$Q_{ij} = \sum_{l}^{n} q_l \ (3r_{il}x_{jl} - |r_l|^2\delta_{ij})$$

$$\Phi_{quadrapole}(\mathbf{r}) = \frac{1}{6}\sum_{i,j} Q_{ij} \frac{(3x_i x_j - \delta_{ij} r^2)}{r^5}$$

$$\Phi_{dipole}(\mathbf{r}) = \int_S \frac{d\mathbf{s'} \bullet \mathbf{P}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} - \int d^3r' \frac{\nabla' \bullet \mathbf{P}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}$$

0.3. **capacitors.**

$$Q = CV$$

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

0.4. **math.**

$$|\mathbf{r}-\mathbf{r'}| = \sqrt{r^2 + r'^2 - 2rr'cos(\theta')}$$

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r'}|^n}\right) = -\frac{n}{|\mathbf{r}-\mathbf{r'}|^{n+2}}(\mathbf{r}-\mathbf{r'}) \quad n >= 1$$

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r'}|}\right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r'})$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

0.5. **Coordiates.**

$x = rsin(\theta)cos(\phi) \quad y = rsin(\theta)sin(\phi) \quad z = rcos(\theta)$

$\theta \in [0,\pi] \quad \phi \in [0,2\pi]$

$x = s\ cos(\phi) \quad y = s\ sin(\phi) \quad z = z$

$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}} \ ; d\tau = dx\ dy\ dz$

$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + rsin(\theta)\hat{\phi} \ ; d\tau = r^2 sin(\theta) dr\ d\theta\ d\phi$

$d\mathbf{l} = ds\hat{\mathbf{s}} + sd\phi\hat{\phi} + dz\hat{\mathbf{z}} \ ; d\tau = s\ ds\ d\phi\ dz$

$$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{rsin(\theta)}\frac{\partial t}{\partial \phi}\hat{\phi}$$

$$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\nabla \bullet \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \bullet \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{rsin(\theta)}\frac{\partial}{\partial \theta}(sin(\theta)v_\theta) + \frac{1}{rsin(\theta)}\frac{\partial v_\phi}{\partial \phi}$$

$$\nabla \bullet \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial t}{\partial r})\frac{1}{r^2 sin(\theta)}\frac{\partial}{\partial \theta}(sin(\theta)\frac{\partial t}{\partial \theta}) + \frac{1}{r^2 sin^2(\theta)}\frac{\partial^2 t}{\partial \phi^2}$$

$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial t}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

0.6. **Solutions to Laplace Equation. Cartesian**

$$X''(x) = \alpha_1 \quad Y''(y) = \alpha_2 \quad Z''(z) = \alpha_2 \quad \alpha_1 + \alpha_2 + \alpha_3 = 0$$

Solve the equation which has some non-zero potential B.C. last.

$$\Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{m,n,k}(X_m(x)Y_n(y)Z_k(z))$$

Spherical (azimuthal symmetry)

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos(\theta))$$

Cylindrical (no z dependence)

$$\Phi(r,\phi) = C_0 + D_0 ln(r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n cos(n\phi) + B_n sin(n\phi))$$

0.7. trig.

$$sin(u \pm v) = sin(u)cos(v) \pm cos(u)sin(v)$$
$$cos(u \pm v) = cos(u)cos(v) \mp sin(u)sin(v)$$
$$sin(2u) = 2sin(u)cos(u)$$
$$cos(2u) = cos^2(u) - sin^2(u) = 1 - 2sin^2(u)$$
$$sin^2(u) = \frac{1 - cos(2u)}{2}$$
$$cos^2(u) = \frac{1 + cos(2u)}{2}$$
$$sin(u) + sin(v) = 2sin(\frac{u+v}{2})cos(\frac{u-v}{2})$$
$$sin(u) - sin(v) = 2cos(\frac{u+v}{2})sin(\frac{u-v}{2})$$
$$cos(u) + cos(v) = 2cos(\frac{u+v}{2})cos(\frac{u-v}{2})$$
$$cos(u) - cos(v) = -2sin(\frac{u+v}{2})sin(\frac{u-v}{2})$$
$$sin(u)sin(v) = \frac{1}{2}(cos(u-v) - cos(u+v))$$
$$cos(u)cos(v) = \frac{1}{2}(cos(u-v) - cos(u+v))$$
$$sin(u)cos(v) = \frac{1}{2}(sin(u+v) - sin(u-v))$$
$$cos(u)sin(v) = \frac{1}{2}(sin(u+v) - sin(u-v))$$

2

1. FOUIER

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(cos(\theta))P_{l'}(cos(\theta))sin(\theta)d\theta = \frac{2}{2l+1} \forall l = l'$$
$$\int_0^{2\pi} cos(a\phi)cos(b\phi) = 0 \ \forall a \neq b \ \ and = \pi \ \forall \ a = b$$
$$\int_0^{2\pi} sin(a\phi)sin(b\phi) = 0 \ \forall a \neq b \ \ and = \pi \ \forall \ a = b$$

2. PROBLEM 1

A conducting sphere of radius R is connect to a battery which keeps it at a constant penitential ϕ_o, relative to a reference point at infinity (i.e. $\phi \to 0$ as r$\to \infty$)

a) What is the total amount of charge that the batter must deposit on the conducting sphere, to keep it at potential ϕ_o?

We know that conductors naturally arrange themselves to be kept at a constant potential. For this reason, there is not need for any charge to be existing on the sphere, if $\phi = 0$. If $\phi \neq 0$ then there would be some charge needed. We can see this by requiring the potential at the surface of the sphere be

$$\phi_o(r = R, \theta, \phi) = \phi_o = \frac{q_1}{R}$$

which is that for a point charge, because if we are outside the sphere, it appears to be a point charge. Rearranging we see

$$q_1 = \phi_o R$$

This can be interpreted as a having a image charge at the center of the sphere.

b) A point charge q is places a distance d from the center of the sphere, where d>R. Now what is the total amount of charge on the conducting sphere?

Well now we have the condition that V(r<R,θ,ϕ)=ϕ_o, but the point charge disrupts this. We can consider the image situation again where we have a point charge q_1 at the center of the sphere like before, but adding another charge q' a distance b from the center of the sphere, where b<R. The hope is that the image charge at the center will keep the sphere at the potential ϕ while the second point charge at b will cancel the potential that the point charge creates. To determine this charge we write the potential at any point outside the sphere

$$\Phi(\mathbf{r}) = \frac{q_1}{|\mathbf{r}|} + \frac{q'}{|\mathbf{r} - b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r} - d\hat{\mathbf{z}}|} \tag{1}$$

$$\Phi(\mathbf{r}) = \frac{\phi_o R}{\sqrt{r^2}} + \frac{q'}{\sqrt{r^2 - b^2 - 2rbcos(\theta)}} + \frac{q}{\sqrt{r^2 - d^2 - 2rdcos(\theta)}} \tag{2}$$

Now we require that the $\Phi(\mathbf{R}) = \phi_o \ \forall\theta$. This leads to an equation which must hold for all angle and you can solve for q' and b. Knowing q_1 and q' we know that the total charge on the sphere is

$$Q_{total} = q_1 + q' \tag{3}$$

What is the force of attracting between q and the conducting sphere? Is it attractive or repulsive? To find the force one can simple use the image situation, finding the force q applies on q_o and q', and taking the sum of those two forces at the total force on the conducting sphere.

$$\mathbf{F}_{\textbf{net sphere}} = q\left(\frac{q_1}{|\mathbf{r_{q_o}} - \mathbf{r_q}|^3}(\mathbf{r_{q_o}} - \mathbf{r_q}) + \frac{q'}{|\mathbf{r_{q'}} - \mathbf{r_q}|^3}(\mathbf{r_{q'}} - \mathbf{r_q})\right) \tag{4}$$

Now to know if the force is attractive or repulsive we must know what the initial potential ϕ_o is. Once we know this, we could determine q_1 which would then allow us to use the above equation to find the magnitude of the net force.

d) Suppose that a cavity exists in the interior of the conducting sphere, and a charge Q is inside the cavity. Now what is the force on q outside?

I think this causes some issues, because my initial point charge was located at the center of the sphere, and now there is the cavity, and the Q is placed on the location of my point charge. I think I could still use the image charge I place at the center. So then I would just add one more term into the equation above, representing that of the Q.

3. PROBLEM 2

Two concentric spherical shells of radii R_1 and R_2, with $R_1 < R_2$, are fixed with the following values of the electrostatic potential:

$$\Phi(R_1, \theta, \phi) = \Phi_1 cos(\theta)$$
$$\Phi(R_2, \theta, \phi) = \Phi_2$$

where Φ_1 and Φ_2 are constants. Let $\Phi \to \infty$.

a) Find the electrostatic potential $\Phi(r, \theta, \phi)$ for r< R_1 (inside the inner shell).

Clearly there is spherical symmetry so we will attempt to match the boundary conditions with the general solution

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos(\theta))$$

When we are inside the inner sphere, we have to be wary of r$\to$0. We see that the we will require that B_l=0 in order to stop this, and so we are left with

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty}(A_l r^l)P_l(cos(\theta))$$

Now we not that we will require that the potential have an overall θ dependence of $cos(\theta)$ for r=R_1. This then suggests that the only l term that survives our sum is $l=1$ which corresponds to the legendar polynomial

$$P_1(cos(\theta)) = cos(\theta)$$

Finally we are left with

$$\Phi(r,\theta) = A_1\ r\ cos(\theta)$$

Now applying the condition that at $\Phi(R_1,\theta,\phi) = \Phi_1 cos(\theta)$ we can solve for A_1 and we get

$$\Phi(r,\theta,\phi) = \frac{\Phi_1}{R_1}\ r\ cos(\theta) \quad \forall\ r < R_1$$

b) Find the electrostatic potential $\Phi(r,\theta,\phi)$ for r> R_2 (outside the inner shell).

Clearly we will use the same general solution

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos(\theta))$$

When we are inside the outside the sphere, we have to be wary of r→ ∞. We see that the we will require that A_l=0 in order to stop this, and so we are left with

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(B_l r^{-(l+1)})P_l(cos(\theta))$$

Now we not that we will require that the potential have no overall θ dependence at for r=R_2. This then suggests that the only l term that survives our sum is $l=0$ which corresponds to the legendra polynomial

$$P_0(cos(\theta)) = 1$$

Finally we are left with

$$\Phi(r,\theta) = A_0\frac{1}{r}$$

Now applying the condition that at $\Phi(R_1,\theta,\phi) = \Phi_2)$ we can solve for A_0 and we get

$$\Phi(r,\theta,\phi) = \Phi_2 R_2\frac{1}{r} \quad \forall\ r > R_2$$

c) Find the electrostatic potential $\Phi(r,\theta,\phi)$ for $R_1 < r < R_2$ (between the shells).

Clearly we will use the same general solution

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos(\theta))$$

In-between the two spheres we do not have to worry about the r dependence going to infinity anywhere. We do know thought that for r = R_1 there should be a θ dependance of just $cos(\theta)$ and at r = R_2 there should be no dependence. This suggests that the only two l values that could possibly exist are $l = 0,1$ which corresponds to the legendra polynomials discussed in a) and b). This leaves us with the general solution

$$\Phi(r,\theta,\phi) = (A_0 r^0 + B_0 r^{-(0+1)})P_0(cos(\theta)) + (A_1 r^1 + B_1 r^{-(1+1)})P_1(cos(\theta))$$

$$\Phi(r,\theta,\phi) = (A_0 + \frac{B_0}{r}) + (A_1 r + \frac{B_1}{r^2})cos(\theta)$$

Now we must ensure that at r=R_2 there is no theta dependence to meat the boundary conditions,this then leads us to see that for the seconds term's coefficient must be zero when r=R_2, that being

$$A_1 R_2 + \frac{B_1}{R_2^2} = 0$$

or

$$A_1 R_2 = -\frac{B_1}{R_2^2}$$

so

$$A_1 = -\frac{B_1}{R_2^3}$$

putting this back into our general solution

$$\Phi(r,\theta,\phi) = (A_0 + \frac{B_0}{r}) + (-\frac{B_1}{R_2^3}r + \frac{B_1}{r^2})cos(\theta)$$

Now we must ensure that at r=R_1 there is a theta dependence to meat the boundary conditions,this then leads us to see that for the first term must be zero when r=R_1, that being

$$A_0 + \frac{B_0}{R_1} = 0$$

leading to

$$A_0 = -\frac{B_0}{R_1}$$

Our solution is now

$$\Phi(r,\theta,\phi) = (-\frac{B_0}{R_1} + \frac{B_0}{r}) + (-\frac{B_1}{R_2^3} r + \frac{B_1}{r^2}) cos(\theta)$$

Now we may use the boundary conditions again to see that

$$\Phi(R_1,\theta,\phi) = (-\frac{B_0}{R_1} + \frac{B_0}{R_1}) + (-\frac{B_1}{R_2^3} R_1 + \frac{B_1}{R_1^2}) cos(\theta) = \Phi_1 cos(\theta)$$

the first term is zero by construction in the previous steps, and so we know that

$$-\frac{B_1}{R_2^3} R_1 + \frac{B_1}{R_1^2} = \Phi_1$$

By two steps of algebra

$$B_1 = \Phi_1 \frac{R_1^2 R_2^3}{R_3^3 - R_1^3}$$

Our solution is now

$$\Phi(r,\theta,\phi) = (-\frac{B_0}{R_1} + \frac{B_0}{r}) + \Phi_1 \frac{R_1^2 R_2^3}{R_3^3 - R_1^3} (\frac{1}{r^2} - \frac{r}{R_2^3}) cos(\theta)$$

Now applying the B.C. again

$$\Phi(R_2,\theta,\phi) = (-\frac{B_0}{R_1} + \frac{B_0}{R_2}) + \Phi_1 \frac{R_1^2 R_2^3}{R_2^3 - R_1^3} (\frac{1}{R_2^2} - \frac{R_2}{R_2^3}) cos(\theta) = \Phi_2$$

the second term is zero by construction, and we find

$$-\frac{B_0}{R_1} + \frac{B_0}{R_2} = \Phi_2$$

so

$$B_0 = \Phi_2 \frac{R_1 R_2}{R_1 - R_2}$$

and so

$$\Phi(r,\theta,\phi) = \Phi_2 \frac{R_1 R_2}{R_1 - R_2} (\frac{1}{r} - \frac{1}{R_1}) + \Phi_1 \frac{R_1^2 R_2^3}{R_2^3 - R_1^3} (\frac{1}{r^2} - \frac{r}{R_2^3}) cos(\theta)$$

c) Find the surface charge $\sigma(\theta,\phi)$ on the shells at r=R_1 and r=R_2

Now we know that the surface charge density σ over a surface is given by

$$\sigma = \frac{1}{4\pi} \hat{\mathbf{n}}(\mathbf{E_r} - \mathbf{E_l})$$

and so we will need the electric field everywhere in space, which will require us take the negative gradient our solution for Φ from a), b) and c). In the interest of time I will not perform the calculation explicitly, but I would take only the radial comment of the gradient, because we will be dotting that with the normal direction, which is radially outward.

4. Problem 3

A thin circular disk of radius R, lying in the xy plane and centered at the origin, has on it a fixed surface charge density given by

$$\sigma(s,\phi) = A\, s\, sin(2\phi)$$

where r and ϕ are the usual polar coordinates in the xy plane.
Compute the electrostatic potential of this disk up through the electric quadruple term. Express your answer in spherical coordinates.

We will clearly be using cylindrical coordinates, and the equation

$$\Phi(\mathbf{r}) = \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Now we must determine what $\rho((r'))$ must be, because we are given a surface charge density. We also note that the surface is finite, and so we will expect to see a step function. We want the charge to exist only at z=0 and so we will have a $\delta(z)$. The charge density is then given by

$$\rho(s',\phi',z') = \sigma(s',\phi',z')\delta(z')\theta(R - s') = As' sin(2\phi')\delta(z')\theta(R - s')$$

now we insert this into the potential equation and use the cylindrical $d^3 r'$

$$\Phi(\mathbf{r}) = \int \frac{s'\, ds'\, d\phi'\, dz' As' sin(2\phi')\delta(z')\theta(R - s')}{|\mathbf{r} - \mathbf{r}'|}$$

What is the $|\mathbf{r}-\mathbf{r}'|$ term though? Well we are seeing the potential at a location **r**, which is along the z axis, and we can choose to center our coordinates at the center of the disc. This then says that the **r**=z$\hat{\mathbf{z}}$. Now the potential coming from arbitrary point $\mathbf{r}'$ is going to have , and so its vector form is $\mathbf{r}'$=s'$\hat{\mathbf{s}}$+z'$\hat{\mathbf{z}}$. It should be noted here that all of the angle information is contained in these two components, the $\hat{\mathbf{s}}$ contains an angular dependence. So when we take the magnitude of the difference we will get

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{s'^2 + (z-z')^2}$$

now we have

$$\Phi(\mathbf{r}) = \int ds'\, d\phi'\, dz' \frac{As'^2 sin(2\phi')\delta(z')\theta(R-s')}{\sqrt{s'^2+(z-z')^2}}$$

integrating over the z variable will require that z'=0 leaving

$$\Phi(\mathbf{r}) = \int ds'\, d\phi' \frac{As'^2 sin(2\phi')\theta(R-s')}{\sqrt{s'^2+z^2}}$$

now we write the explicit limits

$$\Phi(\mathbf{r}) = A\int_0^{2\pi} d\phi' sin(2\phi') \int_0^{\infty} ds' \frac{s'^2\theta(R-s')}{\sqrt{s'^2+z^2}}$$

Now we note that the ϕ integral is apparently zero, which would indicate that the potential is zero. That seems a little odd, but I cannot find a mistake.

5. Problem 4 (Griffiths 3.36)

Two long straight wires, carrying opposite uniform line charges $+/-\lambda$, are situated on either side of a long conducting cylinder. The cylinder (which carries no net charge) has a radius R, and the wires are a distance a from the axis. Find the potential at a point r.

We first must indicate that there will be some polarization in the conductor, and positive charges are pulled to the left, and negative are pulled to the right with magnitudes equal to $+/-\lambda$. This would indicate that one could create two image charges, which are themselves infinite line charges parallel to those in the problem, located at a distance b from the origin of the center cylinder.

One can now indicate the result obtained in class that for an infinite line charge the potential is given by

$$\Phi(\mathbf{r}) = -2\lambda\, ln(s)$$

With this we may simple say that the potential at some location **r** is

$$\Phi(\mathbf{r}) = -2\lambda(-ln(s_1) + ln(s_2 - ln(s_3) + ln(s_4))$$

where s_i is the distance to **r** from the point charges. To find these distances one just does a shift in coordinates.

6. Problem 5 (Griffiths 3.39)

A long cylindrical shell of radius R carries a uniform surface charge σ_0 on the upper half and an opposite charge $-\sigma_0$ on the lower half. Find the electric potential inside and outside the cylinder. There is clearly no charges in this problem so we may solve Laplaces equation to find the potential. Also there is clearly going to be no z dependence and so we may use our general solution to to Laplaces equation in cylindrical coordinates with no z dependence given by

$$\Phi(r,\phi) = C_0 + D_0 ln(r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n cos(n\phi) + B_n sin(n\phi))$$

We will begin with the inside. We require that as r→0 that Φ be finite, and so this allows us to determine that $D_0 = D_n = 0$ because ln(r) and r^{-n} tends to $-\infty$ as r→0. This reduces our solution to

$$\Phi(r,\phi) = C_0 + \sum_{n=1}^{\infty} C_n r^n (A_n cos(n\phi) + B_n sin(n\phi))$$

Now for outside, coefficients indicated with a tic mark, we will require that Φ=0 at ∞ and so C_o'=D_0'=C_n' for this to occur (ln() and r^n tend to infinity for r → ∞), and so we are left with

$$\Phi(r,\phi) = \sum_{n=1}^{\infty} D_n' r^{-n}(A_n' cos(n\phi) + B_n' sin(n\phi))$$

Now we will also require that the two potentials agree when r=R, and so that leads to

$$\sum_{n=1}^{\infty} D_n' R^{-n}(A_n' cos(n\phi) + B_n' sin(n\phi)) = C_0 + \sum_{n=1}^{\infty} C_n R^n (A_n cos(n\phi) + B_n sin(n\phi))$$

requiring that C_o=0 and we get

$$\sum_{n=1}^{\infty} D_n' R^{-n}(A_n' cos(n\phi) + B_n' sin(n\phi)) = \sum_{n=1}^{\infty} C_n R^n (A_n cos(n\phi) + B_n sin(n\phi))$$

now for each k, this equality must hold and so we have two equations

$$E'_n R^{-n} cos(n\phi) = F_n R^n cos(n\phi)$$

$$G'_n R^{-n} sin(n\phi) = H_n R^n sin(n\phi)$$

where we have absorbed all the constants into a single constant, canceling the angular dependence and solving we find

$$E'_n = F_n R^{2n}$$

$$G'_n = H_n R^{2n}$$

Now we also know that

$$\sigma = \frac{1}{4\pi}\hat{\mathbf{n}} \bullet (\mathbf{E_r} - \mathbf{E_l}) = \frac{1}{4\pi}(\frac{\partial\Phi_{in}}{\partial r} - \frac{\partial\Phi_{out}}{\partial r})$$

where we have used the fact that the normal direction is the radial component. This sets another restriction on our problem, that being

$$\sum_{n=1}^{\infty} -nR^{-n-1}(E'_n cos(n\phi) + G'_n sin(n\phi)) - \sum_{n=1}^{\infty} nR^{n-1}(F_n cos(n\phi) + H_n sin(n\phi)) = 4\pi\sigma(\theta)$$

by taking the derivatives and applying our redefinition of coefficients. Now substituting our coefficients we just solved for we find

$$\sum_{n=1}^{\infty} -nR^{-n-1}(F_n R^{2n} cos(n\phi) + H_n R^{2n} sin(n\phi)) - \sum_{n=1}^{\infty} nR^{n-1}(F_n cos(n\phi) + H_n sin(n\phi)) = 4\pi\sigma(\theta)$$

Now by combining these terms and simplifying

$$\sum_{n=1}^{\infty} 2nR^{n-1}(F_n cos(n\phi) + H_n sin(n\phi)) = 4\pi\sigma(\theta)$$

We note that sigma is a discontinuous function with period $2pi$. At this point one must see that this is just a fourier series for σ, whose coefficients can be used by the fourier trick, which is to multiply both sides of the equation by the integral of $cos(l\phi)$ from 0 to 2π to find the coefficients of the $cos(n\phi)$ term, and to find the coefficnats of the $sin(n\phi)$ term we do the same, but with $sin(l\phi)$. That is to say

$$\sum_{n=1}^{\infty}\int_0^{2\pi} cos(l\phi) 2nR^{n-1}(F_n cos(n\phi) + H_n sin(n\phi)) = \int_0^{2\pi} cos(l\phi) = \int_0^{\pi} cos(l\phi)\sigma_0 - \int_{\pi}^{2\pi} cos(l\phi)\sigma_0$$

for the cos term, and

$$\sum_{n=1}^{\infty}\int_0^{2\pi} sin(l\phi) 2nR^{n-1}(F_n cos(n\phi) + H_n sin(n\phi)) = \int_0^{2\pi} sin(l\phi) = \int_0^{\pi} sin(l\phi)\sigma_0 - \int_{\pi}^{2\pi} sin(l\phi)\sigma_0$$

for the sin term. This might look terrible, but there is some hope. We know that

$$\int_0^{2\pi} sin(a\phi)cos(b\phi)d\phi = 0$$

which sends two of the integrals to zero. The we also note that

$$\int_0^{2\pi} cos(a\phi)cos(b\phi) = 0 \;\forall a \neq b \;\; and = \pi \;\forall\; a = b$$

$$\int_0^{2\pi} sin(a\phi)sin(b\phi) = 0 \;\forall a \neq b \;\; and = \pi \;\forall\; a = b$$

which acts like a delta function, and turns all elements into the sum equal to zero except for l=n. This leaves us then with the equations

$$2nlR^{l-1}\pi F_l = \int_0^{\pi} cos(l\phi)\sigma_0 - \int_{\pi}^{2\pi} cos(l\phi)\sigma_0$$

for the cos term, and

$$2nlR^{l-1}\pi H_l = \int_0^{\pi} sin(l\phi)\sigma_0 - \int_{\pi}^{2\pi} sin(l\phi)\sigma_0$$

Evaluating these integrals is trivial, the first is zero, and the second is $\sigma_0(2 - cos(l\pi))/l^2$. This leads to $F_i = 0$ and

$$H_l = \frac{\sigma_0(2 - cos(l\pi))}{2l^2 R^{l-1}\pi}$$

leading to the condition that all even l coefficients are zero, and that odd ones are

$$H_l = \frac{2\sigma_0}{l^2 R^{l-1}\pi}$$

Now we can use this information go put these coefficients into our expansion to find final answers by using the expansion I found way at the beginning. Note that you must go back to find the relationship between the primed coefficients (outside) and the unprimed coefficients (inside) and use those where they are appropriate.

7. Problem 6 (Griffiths 3.40)

A thin insulating rod, running from z=-a to z=+a, carries the indicated line charges. In each case, find the leading term in the multipole expansion of the potential.

a) $\lambda = kcos(\frac{\pi z}{2a})$

We will always start with the monopole, then dipole, then quadruple, because we are only seeking the leading term, and these are decreasing in overall effectiveness, so this could save use time, rather than calculating the quadruple, then dipole, then monopole moment.

The general form of the multipole expansion for distances far away is

$$\Phi(\mathbf{r}) = \int \rho(\mathbf{r}') \sum_{n=0}^{\infty} P_n(cos(\gamma)) \frac{r'^n}{r^{n+1}} d^3r'$$

Now one can write these linear charge densities as volume charge densities by putting two delta functions, one requiring x=0 and one requiring y=0. We first calculate the monopole term which corresponds to

$$\Phi_{monopole} = \int \rho(\mathbf{r}') P_0(cos(\gamma)) \frac{r'^0}{r^{0+1}} d^3r'$$

which, using that $P_0 = 1$ and plugging in our charge density, reduces to

$$\Phi_{monopole} = \frac{1}{r} \int kcos(\frac{\pi z'}{2a}) \delta(x')\delta(y') d^3r'$$

which becomes

$$\Phi_{monopole} = \frac{1}{r} \int_{-a}^{+a} kcos(\frac{\pi z'}{2a}) dz'$$

doing this integral reduces to

$$\Phi_{monopole} = \frac{4ak}{\pi r}$$

b) $\lambda = kcos(\frac{\pi z}{a})$

Now all the same arguments hold as before but the integral becomes

$$\Phi_{monopole} = \frac{1}{r} \int_{-a}^{+a} kcos(\frac{\pi z'}{a}) dz'$$

which is zero because this is over one period. Therefore we must look for the dipole moment which is the second term in the sum which is

$$\Phi_{dipole} = \int \rho(\mathbf{r}') P_1(cos(\gamma)) \frac{r'^1}{r^{1+1}} d^3r'$$

now using that $P_1(cos(\gamma)) = cos(\theta)$ because since we are not in spherical coordinates out thetas are not already caught up together (WHY?), we see that

$$\Phi_{dipole} = \frac{cos(\theta)}{r^2} \int kcos(\frac{\pi z'}{a}) \delta(x')\delta(y') r' d^3r'$$

here we must not that the r' that is in the definition, is the magnitude of the position vector pointing to the charge at $\mathbf{r}'$, and so for our charge this would simple be z', because all the charge lies on the z axis, and so we get

$$\Phi_{dipole} = \frac{kcos(\theta)}{r^2} \int z' cos(\frac{\pi z'}{a}) \delta(x')\delta(y') d^3r'$$

integrating by parts one finds that

$$\Phi_{dipole}(r,\theta) = (\frac{2a^2k}{\pi}) \frac{1}{r^2} cos(\theta)$$

c) $\lambda = kcos(\pi z/a)$

The rest is just simple calculation, but one will find that the monopole, and dipole terms are zero, and the only one that lasts is the third term in the series, and one will find

$$\Phi_{quadrapole}(r,\theta) = \frac{-4a^3k}{\pi^2} \frac{(3cos^2(\theta) - 1)}{2r^3}$$

8. PROBLEM 7 GIFFITHS 3.28

in Ex. 3.9 we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = kcos(\theta)$**.**
a) Calculate the dipole moment of this charge distribution. We know the dipole moment for a continuous distribution is

$$|\mathbf{p}| = \int d^3r|\mathbf{r}|\rho(\mathbf{r})$$

and we will deal with the direction later, but for now we will put in a $\hat{\mathbf{d}}$ to remember its a vector. We can rewrite our charge as a volume charge density and plug into the integral

$$\mathbf{p} = \hat{\mathbf{d}} \int r\ kcos(\theta)\delta(r-R)\ r^2\ sin(\theta)drd\theta d\phi$$

$$\mathbf{p} = \hat{\mathbf{d}}k \int r^3\ cos(\theta)\delta(r-R)\ sin(\theta)drd\theta d\phi$$

the r integral will just make r=R because of the delta function, and the phi is just 2π.

$$\mathbf{p} = \hat{\mathbf{d}}2\pi kR^3 \int_0^\pi sin(\theta)d\theta$$

which is an easy integral resulting in 2 and so we get

$$\mathbf{p} = 4\pi kR^3\hat{\mathbf{d}}$$

now one must determine the direction, by the symmetry of the geometry, it is clear that the direction is going to be in the z direction. To see this draw where the charges are on the sphere at the extreme values of sin(θ) and you will see that the majority of the charge will lie on the poles, like point charges.