Example 3 Diff. eqs. (25 min)

$$\boxed{\begin{array}{ll} f''(t) - a^2 f(t) = A\sin(\omega t) & , \text{ D.E} \\ t \geq 0 & , \text{ R.o I} \\ f(0) = f_0 \;,\; f'(0) = f_1 & , \text{ I.C} \end{array}}$$

1) $\hat{\mathcal{L}}$ both sides

$$LHS = \hat{\mathcal{L}}_{t\to s}(f'' - a^2 f) = s^2\bar{f}(s) - sf_0 - f_1 - a^2\bar{f}(s)$$

RHS $\hat{\mathcal{L}}_{t\to s}(A\sin(\omega t)) = A\int_0^\infty \sin\omega t\, e^{-st}\,dt =$

$\boxed{\begin{array}{l} Re\{s\} = c > 0 \\ \lim_{t\to\infty}(e^{-st}) = 0 \end{array}}$

$$= \frac{A}{2i}\left[\int_0^\infty e^{-(s-i\omega)t}\,dt - \int_0^\infty e^{-(s+i\omega)t}\,dt\right] =$$

$$= \frac{A}{2i}\left(-\frac{e^{-(s-i\omega)t}}{s-i\omega} + \frac{e^{-(s+i\omega)t}}{(s+i\omega)}\right)\Bigg|_0^\infty = \frac{A}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right)$$

Don't combine

$$(s^2 - a^2)\bar{f}(s) - sf_0 - f_1 = \frac{A}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right)$$

2) Solve for $\bar{f}(s)$

$$\bar{f}(s) = \frac{sf_0}{s^2-a^2} + \frac{f_1}{s^2-a^2} + \frac{A}{2i}\left(\frac{1}{s^2-a^2}\frac{1}{s-i\omega} - \frac{1}{s^2-a^2}\frac{1}{s+i\omega}\right)$$

3) Inverse $\bar{f}(s)$

$$f(t) = \sum Res\{\bar{f}(s)e^{st}; s_k\}$$

$$\Rightarrow f(t) = \frac{sf_0e^{st}}{2s} + \frac{sf_0}{2s}\bigg|_{-a} + \frac{f_1e^{st}}{2s}\bigg|_a + \frac{f_1e^{st}}{2s}\bigg|_{-a} +$$

$$+ \frac{A}{2i}\left(\frac{e^{st}}{2s(s-i\omega)} - \frac{e^{st}}{2s(s+i\omega)}\right)\bigg|_a + \frac{A}{2i}\left(\frac{e^{st}}{2s(s-i\omega)} - \frac{e^{st}}{2s(s+i\omega)}\right)\bigg|_{-a} +$$

$$+ \frac{A}{2i}\left(\frac{e^{st}}{s^2-a^2}\bigg|_{i\omega} - \frac{e^{st}}{s^2-a^2}\bigg|_{-i\omega}\right) =$$

$$= f_0 \cosh(at) + \frac{f_1}{a}\sinh(at) + \frac{A}{2i}\left[\frac{1}{2a(a-i\omega)} - \frac{1}{2a(a+i\omega)}\right]e^{at}$$

$$+\frac{A}{2i}\left[\left(\frac{1}{2a(a+i\omega)} - \frac{1}{2a(a-i\omega)}\right)e^{-at}\right] + \frac{A}{2i}\left(-\frac{e^{i\omega t}}{\omega^2+a^2} + \frac{e^{-i\omega t}}{\omega^2+a^2}\right)$$

$$= f_0 \cosh(a) + \frac{f_1}{a}\sinh(at) + \frac{A}{\cancel{2i}}\sinh(a)\frac{\cancel{2i}\omega}{\cancel{2}a(a^2+\omega^2)} - \frac{A}{\omega^2+a^2}\sin\omega t$$

$$= \boxed{f_0\cosh(at) + \left(f_1 + \frac{\omega A}{a^2+\omega^2}\right)\frac{\sinh(at)}{a} - \frac{A}{\omega^2+a^2}\sin\omega t}$$

4) Check.

$$\boxed{f(0) = 0} \quad \checkmark$$

$$f'(t) = af_0\sinh(at) + \left(f_1 + \frac{\omega A}{a^2+\omega^2}\right)\cosh(at) - \frac{A\omega}{a^2+\omega^2}\cos\omega t$$

$$\boxed{f'(0) = f_1} \quad \checkmark$$

$$f''(t) = a^2 f_0\cosh(at) + a\left(f_1 + \frac{\omega A}{a^2+\omega^2}\right)\sinh(at) + \frac{A\omega^2}{a^2+\omega^2}\sin\omega t$$

$$\Rightarrow f''(t) - a^2 f(t) = \frac{A\cancel{(\omega^2+a^2)}}{\cancel{\omega^2+a^2}}\sin\omega t = \boxed{A\sin(\omega t)} \quad \checkmark$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$