How to solve Problem 4

In class next week I will discuss a general procedure for solving these problems, but here I give the general idea, so you can solve this problem. Suppose that you have a second order differential equation:

$$\phi''(x) + p(x)\phi'(x) + q(x)\phi(x) = 0$$

where $p(x)$ and $q(x)$ are known functions. There are two independent solutions for $\phi(x)$, which we call $\phi_1(x)$ and $\phi_2(x)$. Suppose that, by inspection, you find $\phi_1(x)$. Then, to find $\phi_2(x)$ you can assume (without loss of generality)

$$\phi_2(x) = \phi_1(x)\, u(x)$$

where $u(x)$ is an unknown function. Substituting this in the D.E. and expanding, we get

$$0 = \phi_2'' + p\phi_2' + q\phi_2 = (u\phi_1)'' + p(u\phi_1)' + q(u\phi_1)$$

$$= u''\phi_1 + u'(2\phi_1' + p\phi_1) + u\underbrace{(\phi_1'' + p\phi_1' + q\phi_1)}_{\to 0}$$

0 because ϕ_1 is a solution

$$\therefore \quad u''\phi_1 + u'(2\phi_1' + p\phi_1) = 0$$

This is a first order differential equation for $u'(x)$ that can be easily solved:

$$\frac{u''}{u'} = -\frac{2\phi_1'}{\phi_1} - p$$

Integrating both sides:

$$\ln u' = \underbrace{-2\ln\phi_1}_{\ln\frac{1}{\phi_1^2}} - \underbrace{\int^x p(x')\,dx'}_{\text{call } \mathcal{P}(x)} + \overset{\substack{\text{constant} \\ \downarrow}}{c_1}$$

exponentiating:

$$u' = \frac{1}{\phi_1^2} e^{-\mathcal{P}} \underset{e^{c_1}}{\underbrace{c_2}} \Rightarrow u(x) = c_2 \int^x \frac{e^{-\mathcal{P}(x')}}{\phi_1^2(x')} dx' + c_3$$

$$\phi_2 = u\phi_1 = c_2\phi_1(x) \int^x \frac{e^{-\mathcal{P}(x')}}{\phi_1^2(x')} dx' + c_3\phi_1(x)$$

Since c_3 just introduces an amount of ϕ_1 and c_2 scales the rest, we can, without loss of generality, set $c_2 = 1$, $c_3 = 0$.

Then $$\phi_2(x) = \phi_1(x) \int^x \frac{e^{-\mathcal{P}(x')}}{\phi_1^2(x')} dx'$$

Example 1:

Suppose we have the equation

$$\phi'' + p_0\phi' + q_0\phi = 0, \quad p_0, q_0 = \text{constant}$$

When the coefficients are constant, try $\phi = e^{\alpha x}$.

Substituting: $\alpha^2 e^{\alpha x} + p_0 \alpha e^{\alpha x} + q_0 e^{\alpha x} = 0$

$$\alpha^2 + p_0\alpha + q_0 = 0$$

$$\alpha_{1,2} = -\frac{p_0}{2} \pm \frac{1}{2}\sqrt{p_0^2 - 4q_0}$$

So the two independent solutions are $\phi_1 = e^{\alpha_1 x}$, $\phi_2 = e^{\alpha_2 x}$. Suppose, though, we only knew $\phi_1 = e^{\alpha_1 x}$.

We can find ϕ_2 with the procedure

$$\phi_2 = u(x)\underbrace{e^{\alpha_1 x}}_{\phi_1}, \quad \alpha_1 = -\frac{p_0}{2} + \frac{1}{2}\sqrt{p_0^2 - 4q_0}$$

substitute:

$$0 = \left(u e^{\alpha_1 x}\right)'' + p_0\left(u e^{\alpha_1 x}\right)' + q_0 u e^{\alpha_1 x}$$

$$= u'' e^{\alpha_1 x} + u'\left(2\alpha_1 e^{\alpha_1 x} + p_0 e^{\alpha_1 x}\right) + u\underbrace{\left(\alpha_1^2 e^{\alpha_1 x} + p_0\alpha_1 e^{\alpha_1 x} + q_0 e^{\alpha_1 x}\right)}_{\to 0}$$

$$u'' + u'\underbrace{(2\alpha_1 + p_0)}_{\cancel{-p_0} + \sqrt{p_0^2 - 4q_0} + \cancel{p_0}} + u\,\cancel{\left(\alpha_1^2 + p_0\alpha_1 + q_0\right)} = u'' + \sqrt{p_0^2 - 4q_0}\,u' = 0$$

$$u''(x) + \sqrt{p_0^2-4q_0}\, u'(x) = 0$$

if $p_0^2 \neq 4q_0$

$$u'(x) = c_1 e^{-\sqrt{p_0^2-4q_0}\,x}$$

$$u(x) = \underbrace{\frac{-c_1}{\sqrt{p_0^2-4q_0}}}_{c_2} e^{-\sqrt{p_0^2-4q_0}\,x} + c_3$$

$$= \underbrace{\cancel{c_2}}_{\text{set to 1}} e^{-\sqrt{p_0^2-4q_0}\,x} + \underbrace{\cancel{c_3}}_{\text{set to zero}}$$

$$\phi_2(x) = u\phi_1 = \cancel{c_2} e^{-\sqrt{p_0^2-4q_0}\,x} e^{-\frac{p_0}{2}x + \frac{1}{2}\sqrt{p_0^2-4q_0}\,x}$$

$$= e^{\left(-\frac{p_0}{2} - \frac{1}{2}\sqrt{p_0^2-4q_0}\right)x} = e^{\alpha_2 x}$$ ✓✓

in agreement with previous result.

even if $p_0^2 = 4q_0$, and $\alpha_1 = -\frac{p_0}{2}$, this method gives the right answer:

$$u''(x) + \cancel{\sqrt{p_0^2-4q_0}\, u'(x)}^{\,0} = 0$$

$$u'(x) = c_1 \quad , \quad u(x) = \underbrace{\cancel{c_1}}_{\text{set to 1}} x + \underbrace{\cancel{c_2}}_{\text{set to zero}}$$

$$\phi_2 = u\phi_1 = x e^{-\frac{p_0}{2}x}$$

Example 2:

Consider now the equation

$$\phi'' + \frac{a}{x}\phi' + \frac{b}{x^2}\phi = 0 \quad , a, b = \text{constants.}$$

when for all terms the # of derivatives + the exponent of $\frac{1}{x}$ is constant, try $\phi(x) = x^\gamma$

$$\gamma(\gamma-1)\,x^{\gamma-2} + a\gamma\,x^{\gamma-2} + b\,x^{\gamma-2} = 0$$

$$\gamma^2 + (a-1)\gamma + b = 0 \qquad \gamma_{1,2} = \frac{1-a}{2} \pm \frac{1}{2}\sqrt{(1-a)^2 - 4b}$$

so, if $(1-a)^2 \neq 4b$ the two solutions are

$$\phi_1 = x^{\gamma_1} \quad , \quad \phi_2 = x^{\gamma_2}$$

Suppose we only knew $\phi_1 = x^{\gamma_1}$. We again try

$$\phi_2 = u\,\underbrace{x^{\gamma_1}}_{\phi_1}$$

$$0 = (u x^{\gamma_1})'' + \frac{a}{x}(u x^{\gamma_1})' + \frac{b}{x^2}(u x^{\gamma_1})$$

$$= u'' x^{\gamma_1} + u'\left(2\gamma_1 x^{\gamma_1-1} + a x^{\gamma_1-1}\right) + u\left(\gamma_1(\gamma_1-1)x^{\gamma_1-2} + a\gamma_1 x^{\gamma_1-2} + b x^{\gamma_1-2}\right)$$

$$u'' x^{\gamma_1} + u'(2\gamma_1 + a)\,x^{\gamma_1-1} + u\,\cancel{\left(\gamma_1(\gamma_1-1) + a\gamma_1 + b\right)}^{\,0}\,x^{\gamma_1-2} = 0$$

$$\frac{u''}{u'} = -(2\gamma_1 + a)\frac{x^{\gamma_1 - 1}}{x^{\gamma_1}} = -\frac{(2\gamma_1+a)}{x}$$

integrating

$$\ln u'(x) = -(2\gamma_1 + a)\ln x$$

exponentiating

$$u' = c_1 x^{-(2\gamma_1 + a)}$$

Note $\quad 2\gamma_1 + a = 1 - a + \sqrt{(1-a)^2 - 4b} + a = 1 + \sqrt{(1-a)^2-4b}$

- if $(1-a)^2 \neq 4b$

$$u = \underbrace{\frac{c_1}{-(2\gamma_1+a)}}_{c_2 \text{ set to } 1} x^{1-(2\gamma_1+a)} + \underbrace{c_3}_{\text{set to zero}}$$

$$u = x^{1-a-2\gamma_1}, \quad \phi_2 = x^{\gamma_1 + 1 - a - 2\gamma_1} = x^{1-a-\gamma_1} = x^{\overbrace{\frac{1-a}{2} - \sqrt{(1-a)^2-4b}}^{\gamma_2}}$$

$$\phi_2 = x^{\gamma_2}$$

- if $(1-a)^2 = 4b$, $\gamma_1 = \frac{1-a}{2}$

$$u' = c_1 x^{-(1-a+a)} = \frac{c_1}{x}, \quad u = \underbrace{c_1}_{\text{set to } 1}\ln x + \underbrace{c_2}_{\text{set to } 0} = \ln x$$

$$\phi_2 = u\phi_1 = \ln x \; x^{\frac{1-a}{2}}$$