Frobenius method (general case)

Consider the homogeneous equation

$$\phi_h'' + p\phi_h' + q\phi_h = 0,$$

where $p(x)$ might include a simple pole, and $q(x)$ up to a second order pole at a "regular singular point" x_0. That is, these functions can be expanded as Laurent series around x_0 as

$$p(x) = \sum_{i=0}^{\infty} P_i (x-x_0)^{i-1}, \quad q(x) = \sum_{i=0}^{\infty} Q_i (x-x_0)^{i-2}.$$

Of course, such series will be valid only up to ~~the~~ any other singularity that p & q might have, as will also the result for ϕ_h.

To solve the equation, we also propose a series expansion for ϕ_h of the form

$$\phi_h(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\gamma}, \text{ where we can assume } a_0 = 1.$$

The extra power of γ is included so that the series can be Taylor, Laurent, or a series in fractional powers (or even complex ones) associated with a branch point. This "index" γ will be determined from the differential equation.

For shorthand, let us abbreviate $\tau = x - x_0$, so that
$$p = \sum_{i=0}^{\infty} P_i \tau^{i-1}, \quad q = \sum_{i=0}^{\infty} Q_i \tau^{i-2}, \quad \phi_h = \sum_{n=0}^{\infty} a_n \tau^{n+\gamma}$$

Substitution into the differential equation gives
$$\sum_{n=0}^{\infty} a_n (n+\gamma)(n+\gamma-1)\tau^{n+\gamma-2} + \sum_{i=0}^{\infty} P_i \tau^{i-1} \sum_{j=0}^{\infty} a_j (j+\gamma)\tau^{j+\gamma-1}$$

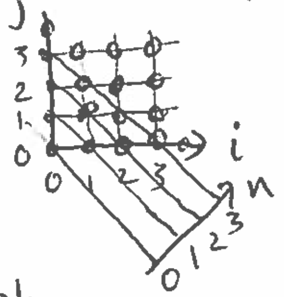
use j instead of n to avoid confusion

$$+ \sum_{i=0}^{\infty} Q_i \tau^{i-2} \sum_{j=0}^{\infty} a_j \tau^{j+\gamma} = 0$$

Reordering and multiplying by $\tau^{2-\gamma}$:
$$\sum_{n=0}^{\infty} a_n (n+\gamma)(n+\gamma-1)\tau^{n} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[P_i a_j (j+\gamma) + Q_i a_j \right] \tau^{i+j} = 0$$

To mix the two terms, let us change index of summation such that $i+j=n \Rightarrow i = n-j$

we can then replace the sum $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$ to $\sum_{n=0}^{\infty}\sum_{j=0}^{n}$



As shown in the picture on the left, we're just changing the order in which we sum.

with this change, we get
$$\sum_{n=0}^{\infty} \left[a_n (n+\gamma)(n+\gamma-1) + \sum_{j=0}^{n} \left(P_{n-j}(j+\gamma) + Q_{n-j} \right) a_j \right] \tau^n = 0$$

This equation must hold for any value of $\tau = x - x_0$. Therefore, each of the coefficients must vanish separately

$$a_n(n+\gamma)(n+\gamma-1)+\sum_{j=0}^{n}\left(P_{n-j}(j+\gamma)+Q_{n-j}\right)a_j=0 \quad \forall n=0,1,\ldots$$

we can rewrite this expression by separating the $n=j$ contribution from the second term, and grouping it with the first:

$$a_n\Big[\underbrace{(n+\gamma)(n+\gamma-1)+P_0(n+\gamma)+Q_0}_{n^2+n(2\gamma+P_0-1)+\gamma^2+(P_0-1)\gamma+Q_0}\Big]+\sum_{j=0}^{n-1}\left(P_{n-j}(j+\gamma)+Q_{n-j}\right)a_j=0$$

Let us start by considering the $n=0$ equation. This equation is called the "indicial" equation because it determines the "index" γ:

$$\cancel{a_0}^{\,1}\left(\gamma^2+(P_0-1)\gamma+Q_0\right)=0 \qquad \text{(the term with the sum does not contribute)}$$

$$\gamma_{\frac{1}{2}}=\frac{1-P_0}{2}\pm\frac{1}{2}\sqrt{(1-P_0)^2-4Q_0}$$

These are the values of γ for the two independent solutions ϕ_1 & ϕ_2.

Now, the equation for $n\geq 1$ gives a recursion relation

0 from indicial eq.

$$a_n\left[n^2+n(2\gamma+P_0-1)+\cancel{(\gamma^2+(P_0-1)\gamma+Q_0)}\right]+\sum_{j=0}^{n-1}\left[P_{n-j}(j+\gamma)+Q_{n-j}\right]a_j=0$$

so

$$a_n=\frac{-\sum_{j=0}^{n-1}\left[P_{n-j}(j+\gamma)+Q_{n-j}\right]a_j}{n(n+2\gamma+P_0-1)}$$

From here, we can determine a_n from all $a_{j<n}$.

Notice that, in principle, this gives the series coefficients for <u>both</u> independent solutions ϕ_1 & ϕ_2, depending on whether we use $\gamma=\gamma_1$ or $\gamma=\gamma_2$. Let us then label the coefficients $a_n^{\{1\}}$ for ϕ_1 & $a_n^{\{2\}}$ for ϕ_2. However, there can be problems for ϕ_2 in some cases. note that $\gamma_1+\gamma_2=1-P_0$, and therefore $2\gamma_{\frac{1}{2}}+P_0-1=2\gamma_{\frac{1}{2}}-(\gamma_1+\gamma_2)=\pm(\gamma_1-\gamma_2)$.

The recursion relation then becomes

$$a_n^{\{\frac{1}{2}\}}=\frac{-\sum_{j=0}^{n-1}\left[P_{n-j}\,(j+\gamma_{\frac{1}{2}})+Q_{n-j}\right]a_j^{\{\frac{1}{2}\}}}{n\left(n\pm(\gamma_1-\gamma_2)\right)}\qquad n\geq 1,\ \text{with } a_0^{\{\frac{1}{2}\}}=1.$$

For the first solution, $n+\gamma_1-\gamma_2$ never vanishes, so there is no problem. On the other hand, for the second solution, the factor $n-(\gamma_1-\gamma_2)$ in the denominator can vanish if $\gamma_1-\gamma_2$ is an integer! In this case, the Frobenius method does not give directly the second solution, so we must use the prescription for finding ϕ_2 in terms of ϕ_1.

Notice that if $\gamma_1=\gamma_2$, the denominator does not vanish, but the solutions for ϕ_1 & ϕ_2 are identical and hence not linearly independent.

In summary, when $\gamma_1-\gamma_2=0,1,2,\ldots$, we must use the prescription to find ϕ_2 from ϕ_1. Otherwise, the Frobenius method does give both solutions.