Partial Fractions and Arcsin Branch Points

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1 PARTIAL FRACTIONS

Direct Computation: A direct computation of the partial fraction coefficients is possible instead of using the lengthy algebraic approach that most of you are familiar with. For instance, assume that you have a fraction \( P(x)/Q(x) \), where \( Q(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \). Assume that \( \deg P \leq \deg Q \). If that is not the case long division of \( P(x) \) by \( Q(x) \) needs to be performed first and the formula described below can be used on the remainder from the long division. Now the expansion of \( P(x)/Q(x) \) as a sum of partial fractions is done by the following formula:

\[
\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)} \frac{1}{(x - \alpha_i)}
\]  

(1)

where \( Q' \) is the derivative of \( Q(x) \). You don’t have to prove this expression.

2 ARCSIN BRANCH POINTS

Assume the analytic extension of the real function \( \arcsin(x) \) to be \( \arcsin(z) \) where \( z = re^{i\theta} \). What would be the singularities of the function \( f(z) = \arcsin(z) \)? To answer this question you can try to do the following few exercises.

- Branch Points of Log: We will start by looking at the singularities of \( \ln(z) \). Express \( z \) as \( z = re^{i\theta} \) and convince yourself that \( \ln(z) \) has a branch point at \( z = 0 \) and that there is an infinite amount of Riemann sheets for that branch point. This is called a logarithmic branch point (vs. algebraic branch points which have finite number of Riemann
sheets.) Now it is also relatively easy to confirm from the same expression that the function has a branch point at infinity. If you can’t see that, try substituting \( z = \frac{1}{z'} \) and let \( z = r e^{i\theta} \) with \( r' \) approaching 0.

- **Expressing Arcsin as a function of Log**: We will have to use some clever algebra to express the \( \text{arcsin}(z) \) in a different form. Use \( \sin(\zeta) = z \), where \( \zeta = \text{arcsin}(z) \). Now using Euler’s formula:

\[
\sin(\zeta) = \frac{e^{i\zeta} + e^{-i\zeta}}{2i}
\]

and solve for \( \zeta \). You should derive the following expression:

\[
\text{arcsin}(z) = -i \ln(i z + (1 - z^2)^{\frac{1}{2}})
\]

Note that you will have to solve a quadratic equation for \( \zeta \). The two solutions of that quadratic equation are hidden behind the branch points at \( z = \pm 1 \) as it can be seen from equation 3.

- **Branch Points of Arcsin**: Now we are ready to find the branch points of \( \text{arcsin}(z) \). As it can be seen from equation 3 there will be algorithmic branch points at \( z = \pm 1 \) because of the fractional \( \frac{1}{2} \) exponent. From the previous section we found out that the natural logarithm has a branch point when the argument is zero. Convince yourself that the argument of the logarithm in equation 3 can not be zero for any value of \( z \). Finally, you can show that there is a branch point at infinity by doing similar substitution like the one described in the previous section: substitute \( z = \frac{1}{z'} \) and let \( z = r e^{i\theta} \) with \( r' \) approaching 0.

- **Conclusion**: The function \( \text{arcsin}(z) \) has branch points at \( z = \pm 1 \) (with two Riemann sheets each) and a logarithmic branch point at infinity.