Self-adjoint Operators

Consider a second order operator

$$\hat{\mathcal{L}} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

For simplicity, assume that p_0, p_1 & p_2 are real.

Consider also two real functions $u(x)$ & $v(x)$.

Let

$$\langle v, \hat{\mathcal{L}} u\rangle = \int_{x_0}^{x_1} v(x)\,\hat{\mathcal{L}} u(x)\,dx$$

$$= \int_{x_0}^{x_1} v\left(p_0 u'' + p_1 u' + p_2 u\right) dx$$

$\hat{\mathcal{L}}$ is self adjoint if, for any u, v:

$$\langle v, \hat{\mathcal{L}} u\rangle = \langle u, \hat{\mathcal{L}} v\rangle + \Big[\quad\Big]\Big|_{x_0}^{x_1}$$

(If u, v, p_0, p_1 & p_2 are allowed to be complex, then the constraint is $\langle v, \hat{\mathcal{L}} u\rangle = \langle u, \hat{\mathcal{L}} v\rangle^* + [\;]\big|_{x_0}^{x_1}$.)

We now check the conditions under which this is true.

$$\langle v, \hat{\mathcal{L}} u\rangle = \int_{x_0}^{x_1} v p_0 u'' \, dx + \int_{x_0}^{x_1} v p_1 u' \, dx + \int_{x_0}^{x_1} v p_2 u \, dx$$

$$\int_{x_0}^{x_1} v p_2 u \, dx = \int_{x_0}^{x_1} u p_2 v \, dx$$

Integrate by parts

$$\int_{x_0}^{x_1} v p_1 u' \, dx = v p_1 u \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u (v p_1)' \, dx = v p_1 u \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u p_1 v' \, dx - \int_{x_0}^{x_1} u p_1' v \, dx$$

Integrate by parts

$$\int_{x_0}^{x_1} v p_0 u'' \, dx = v p_0 u' \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u' (v p_0)' \, dx \underset{\text{integrate by parts again}}{=} v p_0 u' \Big|_{x_0}^{x_1} - (v p_0)' u \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} u (v p_0)'' \, dx$$

$$= \Big[v p_0 u' - v' p_0 u - v p_0' u \Big] \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} u \left(v'' p_0 + 2 v' p_0' + v p_0'' \right) dx$$

So

$$\langle v, \hat{\mathcal{L}} u\rangle = \Big[v p_1 u + v p_0 u' - v' p_0 u - v p_0' u \Big] \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} u \Big[p_0 v'' + (2 p_0' - p_1) v' + (p_0'' - p_1' + p_2) v \Big] dx$$

This equals $[\ ] \Big|_{x_0}^{x_1} + \langle u, \hat{\mathcal{L}} v\rangle = [\ ] \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} u (p_0 v'' + p_1 v' + p_2 v) dx$

if:
- $p_0 = p_0$
- $p_1 = 2 p_0' - p_1$
- $p_2 = p_0'' - p_1' + p_2$

$$\Longrightarrow \boxed{p_0' = p_1}$$

That is, if we let $p_0=\alpha$, $p_2=\beta$,

$\hat{\mathcal{L}}$ is self-adjoint if it can be written as

$$\hat{\mathcal{L}}u=(\alpha u')'+\beta u$$

Can we turn a non-self-adjoint operator into a self-adjoint one? Yes! For example, the ODEs we've been working with have the form

$$\hat{\mathcal{D}}y(x)=y''(x)+p(x)\,y'(x)+q(x)\,y(x)$$

Let $y=\alpha u$

$$\hat{\mathcal{D}}(\alpha u)=\alpha u''+\underbrace{(2\alpha'+p\alpha)}_{=\alpha'}u'+(\alpha''+p\alpha'+q\alpha)u$$

choose α such that

$$2\alpha'+p\alpha=\alpha' \Rightarrow \frac{\alpha'}{\alpha}=-p \Rightarrow \alpha=W(x)=W_0e^{-\mathcal{P}(x)}$$

So α must be chosen as the wronskian!

Note: $\beta=(\alpha''+p\alpha'+q\alpha)=(-p\alpha)'+p\alpha'+q\alpha=(q-p')\alpha$

$(\alpha')'=(-p\alpha)'$

so
$$\hat{\mathcal{D}}y=\hat{\mathcal{L}}u, \text{ where } \alpha=W(x),\ \beta=(q-p')W(x)$$

Using the self-adjoint form to predict the possibility of modes.

Recall: a mode is a solution of $\hat{\mathcal{D}} y = 0$ that vanishes at **both** boundaries.

e.g. for
$$\begin{cases} \text{Dirichlet} & y(x_0) = y(x_1) = 0 \\ \text{Neumann} & y'(x_0) = y'(x_1) = 0 \\ \text{Periodic} & y(x_0) = y(x_1),\ y'(x_0) = y'(x_1). \end{cases}$$

Modes make the solution of boundary value problems problematic.

Let $y = \overset{\text{wronskian}}{\cancel{\alpha}}\, u$, so the equation becomes

$$\hat{\mathcal{D}} y = \hat{\mathcal{L}} u = 0.$$

Now consider

$$\langle u, \hat{\mathcal{L}} u \rangle = \int_{x_0}^{x_1} u\left[(\alpha u')' + \beta u\right] dx = \underbrace{\int_{x_0}^{x_1} u (\alpha u')' dx}_{\text{integrate by parts}} + \int_{x_0}^{x_1} \beta u^2 dx$$

$$= u(\alpha u')\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u' \alpha u' dx + \int_{x_0}^{x_1} \beta u^2 dx$$

$$= \underbrace{\alpha u u'\Big|_{x_0}^{x_1}}_{\text{vanishes if}} + \int_{x_0}^{x_1} \left(\beta u^2 - \alpha (u')^2\right) dx$$

Vanishes if
$u(x_0) = u(x_1) = 0$
$u'(x_0) = u'(x_1) = 0$
$u(x_0) = u(x_1),\ u'(x_0) = u'(x_1),\ \alpha(x_0) = \alpha(x_1)$

note: if u is complex and $\langle u, \hat{\mathcal{L}} u \rangle = \int_{x_0}^{x} u^* \hat{\mathcal{L}} u\, dx$, this expression becomes
$$\int_{x_0}^{x_1} \left(\beta |u|^2 - \alpha |u'|^2\right) dx$$

So, for Dirichlet, Neumann (if $\alpha'(x_0)=w'(x_0)=0$), or periodic (if $\alpha(x_0)=\alpha(x_1)$),

$$\underset{\uparrow\ \text{because } \hat{\mathcal{L}}u=0}{0} = \langle u, \hat{\mathcal{L}} u \rangle = \int_{x_0}^{x_1} \left(\beta |u|^2 - \alpha |u'|^2\right) dx$$

Suppose $\alpha \neq 0\ \forall x \in [x_0, x_1]$, i.e. α keeps its sign (the Wronskian does not vanish in the R.O.I.), then, if $\beta\alpha \leq 0\ \ \forall x \in [x_0, x_1]$ there cannot be cancellation in the integral $\int_{x_0}^{x_1} (\beta|u|^2 - \alpha|u'|^2)dx$ and it <u>cannot</u> vanish. Since to get to the equation above we assume modes, this means that:

$$\boxed{\text{there can be no modes if } \alpha\beta \leq 0 \quad \forall x \in [x_0, x_1]}$$

Example 1:

$y'' - k^2 y = 0$ (already in self-adjoint form), $k>0$

$\alpha = 1,\ \beta = -k^2$, $\alpha\beta = -k^2 \leq 0$ No modes!

note $\phi_{1,2} = e^{\pm kx}$, so one cannot combine ϕ_1 & ϕ_2 to give a solution that vanishes at two points.

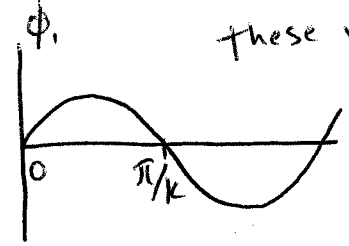
Example 2:

$y'' + k^2 y = 0$, $k>0$

$\alpha = 1,\ \beta = k^2$, $\alpha\beta = k^2 > 0$ <u>May have modes</u>

note

$\phi_{1,2} = \sin kx,\ \cos kx$



these vanish at many points. If we are unlucky enough that $x_1 - x_0 = \frac{\pi}{k} \underset{\substack{\uparrow \\ \text{integer}}}{N}$, then we have modes.